Combinatorics Homework Booklet

A ① $n = 30$ order doesn't matter $\Longrightarrow {}_{30}C_5 = \frac{30!}{5!(30-5)!} = \frac{30!}{5!\,25!}$

$r = 5$

C ② 4th term $\Rightarrow r=3$ in $t_{r+1} = {}_nC_r\, a^{n-r} b^r$ to get t_4

So for $(x-2)^6$...

$n = 6$

$r = 3$

$a = x$

$b = -2$

$t_4 = {}_6C_3\,(x)^{6-3}(-2)^3$

$= 20\,x^3(-8)$

$= -160x^3$

A ③ APPLEPIE

$n = 8$ (b/c 8 letters)

$a = 3$ (3 p's)

$b = 2$ (2 e's)

So $\frac{n!}{a!\,b!} = \frac{8!}{3!\,2!} = 3360$

B ④ $\underbrace{\underline{6} \times \underline{5} \times \underline{4}}_{\text{A to F w/o repeats (6 letters)}} \times \underbrace{\underline{9} \times \underline{8} \times \underline{7}}_{\text{digits 1-9 w/o repeats (9 \#s)}} \times \underline{3} = 181440$

X, Y, or Z

C ⑤ There are 13 different ranks + 4 of each rank. A pair means picking 2 of 4, which can be done for any of the 13 ranks. Then there is 1 more to choose from the remaining 48 (since you can't pick a rank already chosen).

$$13 \times {}_4C_2 \times {}_{48}C_1 = 3744$$

A ⑥ Coach picks 3 out of 10 players and order matters:

$\underline{10} \times \underline{9} \times \underline{8} = 720$ or ${}_{10}P_3 = 720$ or ${}_{10}P_3 = \frac{10!}{(10-3)!} = \frac{10!}{7!}$

A ⑦ 4th term of $(x-2y)^5$

$n = 5$

$r = 3$

$a = x$

$b = -2y$

$t_{r+1} = {}_nC_r\, a^{n-r} b^r$

$t_4 = {}_5C_3\, x^2(-2y)^3$

$= 10\,x^2(-2)^3 y^3$

$= -80x^2y^3$

C ⑧ $_{33}C_5$ use $_nC_r = \frac{n!}{r!(n-r)!}$ $\therefore$ $_{33}C_5 = \frac{33!}{5!(33-5)!}$

$n = 33$ $= \frac{33!}{5!\,28!}$

$r = 5$

C ⑨ 3^{rd} term of $(x-y)^{10}$

$n = 10$ $t_{r+1} = {_nC_r}\, a^{n-r} b^r$

3^{rd} term $\Rightarrow r = 2$ $\therefore t_3 = {_{10}C_2}\,(x)^8(-y)^2$

$a = x$

$b = -y$ $= 45x^8y^2$

C ⑩ Choose 1 pasta of 4 and 1 sauce of 2.

$_4C_1 \times {_2C_1} = 4 \times 2 = 8$ OR $\underset{\text{pasta}}{\underline{4}} \times \underset{\text{sauce}}{\underline{2}} = 8$

C ⑪ $\underline{7} \times \underline{6} \times \underline{5} = 210$ • order matters • no repetitions • 3 picked from 7

B ⑫ 3^{rd} term of $(2x+y)^6$

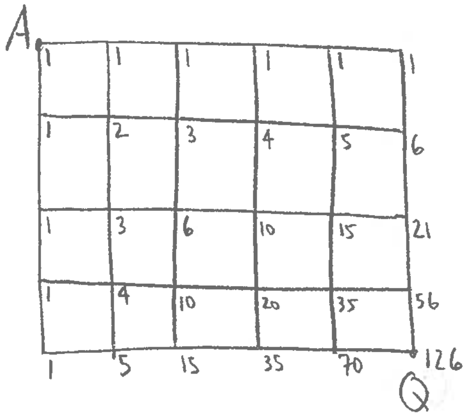
$n = 6$ $t_{r+1} = {_nC_r}\, a^{n-r} b^r$

$r = 2$ $t_3 = {_6C_2}\,(2x)^4(y)^2$

$a = 2x$ $= 15(2)^4x^4y^2$

$b = y$ $= 240x^4y^2$

B ⑬



126 pathways

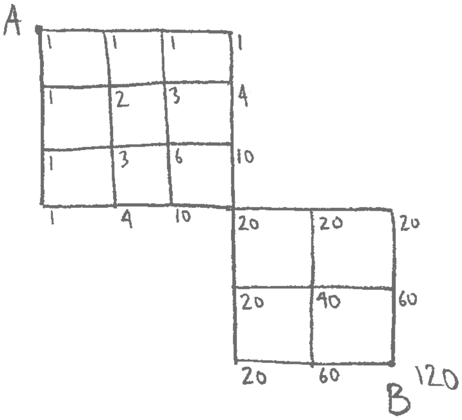
D ⑭ $_nC_r = \frac{n!}{r!(n-r)!}$ so $_nC_2 = \frac{n!}{2!(n-2)!}$

$r = 2$ $= \frac{n(n-1)\cancel{(n-2)!}}{2!\cancel{(n-2)!}}$

$= \frac{1}{2}(n^2 - n)$

(15)

C



$\therefore$ 120 pathways

(16)

D

$$\frac{n(n+1)!}{(n-1)!} = \frac{n(n+1)(n)\cancel{(n-1)!}}{\cancel{(n-1)!}} = n^2(n+1) = n^3+n^2$$

(17) $(2a-3b)^6$ — coefficient of term containing a^4b^2

C

using $t_{r+1} = {}_nC_r\, a^{n-r}b^r \xrightarrow{w/\ n=6} a^{6-r}b^r = a^4b^2 \Rightarrow r=2$

So use
$$\begin{aligned} t_{2+1} &= {}_6C_2\,(2a)^{6-2}(-3b)^2 \\ &= 15(2)^4a^4(-3)^2b^2 \\ &= 15(16)(9)\,a^4b^2 \\ &= 2160\,a^4b^2 \end{aligned}$$

Coefficient of term w/ a^4b^2

(18) $n=49$, $r=6$

B

use ${}_nC_r = \frac{n!}{r!(n-r)!}$ so ${}_{49}C_6 = \frac{49!}{6!(49-6)!} = \frac{49!}{6!\,43!}$

(19) $n=12$ (12 buttons)

B

$a=4$ (4 reds)

$b=4$ (4 greens)

$c=4$ (4 yellows)

$$\frac{12!}{4!\,4!\,4!} = 34650$$

(20) $\underline{9} \times \underline{10} \times \underline{5} = 450$

B

digits 1 to 9 possible; digits 0 to 9 possible; digits 1, 3, 5, 7, 9 possible

(21) $n=5$ $\quad$ use ${}_nC_r = \frac{n!}{r!(n-r)!}$ $\quad$ so ${}_5C_2 = \frac{5!}{2!(5-2)!} = \frac{5!}{2!\,3!}$

C $\quad r=2$

order doesn't matter

(22) 5th term of $(x - \frac{1}{2}y)^7$ $\quad$ use $t_{r+1} = {}_nC_r\, a^{n-r} b^r$

B

$n=7$

$r=4$ (since $5-1=4$)

$a=x$

$b=-\frac{1}{2}y$

$t_{4+1} = {}_7C_4\,(x)^{7-4}\left(-\frac{1}{2}y\right)^4$

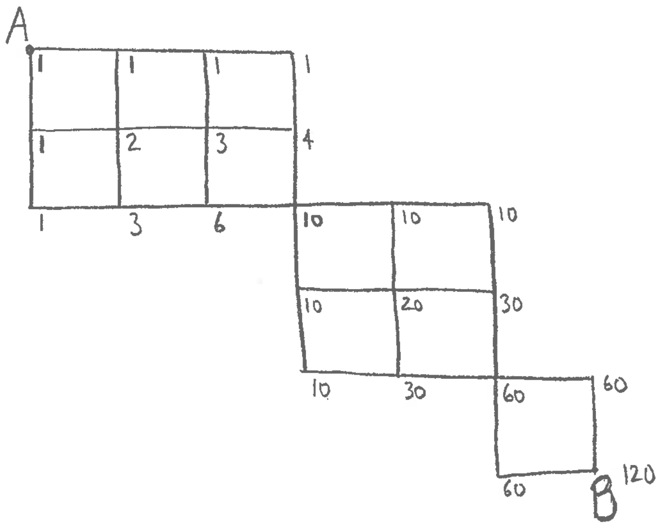
$t_5 = 35\,x^3\left(-\frac{1}{2}\right)^4 y^4$

$= 35\left[\frac{1}{(-2^4)}\right]x^3y^4$

$= \frac{35}{16}\,x^3y^4$

(23)

C



$\therefore$ 120 pathways

Written Questions

(1) Solve algebraically: $\frac{(n-1)!}{(n-3)!} = 30$

$$\frac{(n-1)(n-2)\cancel{(n-3)!}}{\cancel{(n-3)!}} = 30$$

$$(n-1)(n-2) = 30$$

$$n^2 - n - 2n + 2 - 30 = 0$$

$$n^2 - 3n - 28 = 0$$

$$(n-7)(n+4) = 0$$

$\therefore n=7$ or $\cancel{n=-4}$ reject negative roots $\because n \in \mathbb{N}$ for ! to be defined

② $n = 30$

a) $r = 3$ order doesn't matter $\Rightarrow {}_{30}C_3 = 4060$

b) $r = 3$ order does matter $\Rightarrow {}_{30}P_3 = 24\,360$

c) Choose 1 boy of 10 $\Rightarrow {}_{10}C_1 = 10$
choose 2 girls of 20 $\Rightarrow {}_{20}C_2 = 190$

must multiply together b/c of the fundamental Counting Principle (these are NOT two different cases, so we do NOT add them!)

Then $10 \times 190 = 1900$

③ a) choose 2 cars of 4 $\Rightarrow {}_4C_2 = 6$
choose 3 trucks of 6 $\Rightarrow {}_6C_3 = 20$

Same idea ☺

So $6 \times 20 = 120$ ways

b) Choose at least 3 cars to have 5 toys chosen in all gives 3 cases:

- 3 of 4 cars and 2 of 6 trucks: ${}_4C_3 \times {}_6C_2 = 60$
- all 4 cars and 1 of 6 trucks: ${}_4C_4 \times {}_6C_1 = 6$

total: 66 ways

④ $(x-2y)^7$

$n = 7$
$a = x$
$b = -2y$

1st 3 terms: $\sum_{r=0}^{2} {}_7C_r\,(x)^{7-r}(-2y)^r$

$$= {}_7C_0\,x^7(-2y)^0 + {}_7C_1\,x^6(-2y)^1 + {}_7C_2\,x^5(-2y)^2$$

$$= (1)\,x^7(1) + (7)\,x^6(-2y) + (21)x^5(-2)^2(y)^2$$

$$= x^7 + (7)(-2)\,x^6y + (21)(4)\,x^5y^2$$

$$= x^7 - 14x^6y + 84x^5y^2$$

⑤ $\dfrac{n!}{(n-2)!\,4!} = 10 \Rightarrow \dfrac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}\,4\cdot3\cdot2\cdot1} = 10$

$\Rightarrow \dfrac{n(n-1)}{\cancel{24}}\,\cancel{\times 24} = 10 \times 24$ (multiply both sides by $4! = 24$)

$\Rightarrow n^2 - n - 240 = 0$

$\Rightarrow (n-16)(n+15) = 0 \Rightarrow n = 16$, $\cancel{n = -15}$ reject $\because n \in \mathbb{N}$

5/1