Logs & Exponents Homework Booklet — Full Solutions

Multiple Choice

B ① $p = q^r \Leftrightarrow \log_q p = r$ } definition ... note the bases are the same:

(exp form) (log form)

q is the base of $q^r = p$

q ——"—— $\log_q p = r$

helps to remember ☺

D ② $\log_2 7.5 = \dfrac{\log 7.5}{\log 2} \doteq 2.91$

use formula $\boxed{\log_a b = \dfrac{\log_c b}{\log_c a}}$ memorize!

* note choose $c = 10$ for easy calculation on calculator

A ③ $y = 2^{x-3} + 4$

note this function is of the form $y = f(x-3) + 4$, where $f(x) = 2^x$

Since the range of $f(x) = 2^x$ is $f(x) > 0$,

then the range of $y = f(x-3) + 4$ is $y > 4$ $\therefore$ range of $y = 2^{x-3}+4$ is $y > 4$

only affects x (domain) affects y (range)

B ④ $16^{x+1} = 8^{1-x}$

method 1: using exponents...

$(2^4)^{x+1} = (2^3)^{1-x}$

$2^{4(x+1)} = 2^{3(1-x)}$

like bases, equate exp

$\therefore 4(x+1) = 3(1-x)$

$4x + 4 = 3 - 3x$

$7x = -1$

$x = -\frac{1}{7}$

method 2: using logs...

log of both sides

$\log_2(16^{x+1}) = \log_2(8^{1-x})$

$(x+1)\log_2 16 = (1-x)\log_2 8$

$(x+1)\log_2 2^4 = (1-x)\log_2 2^3$

$(x+1)(4) = (1-x)(3)$

$\therefore x = -\frac{1}{7}$

or if all else fails:

$\log(16^{x+1}) = \log(8^{1-x})$

$(x+1)\log 16 = (1-x)\log 8$

convert to decimal

after solving, $x \doteq -0.1429$

* not as convenient if you need answer as a fraction ☺

C ⑤ $\log\left(\dfrac{100x^3}{y}\right) = \log 100 + \log x^3 - \log y$

① addition rule ② subtraction rule

$= 2 + 3\log x - \log y$

Since $\log 100 = \log_{10} 10^2 = 2$

③ exponent rule

① $\log_c a + \log_c b = \log_c ab$

② $\log_c a - \log_c b = \log_c \frac{a}{b}$

③ $\log_c a^b = b\log_c a$

Can also think of ① + ② as product + quotient rules, respectively (why?)

D ⑥ $\log_3(x+4) + \log_3(6-x) = 2$

$\log_3[(x+4)(6-x)] = \log_3 3^2$

addition rule

chose this base b/c same on LS of equation

(like bases (on logs) ∴ equate what the log is being applied to)

$\therefore (x+4)(6-x) = 3^2$

$6x - x^2 + 24 - 4x = 9$

$x^2 - 2x - 15 = 0$

$(x-5)(x+3) = 0$

(factor... think what 2 factors of -15 sum to -2)

OR, use quadratic eqⁿ:

for $ax^2+bx+c=0$,

$x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

here $a=1$, $b=-2$, $c=-15$

By factoring or using quadratic eqⁿ, we find $x=5$ & $x=-3$

BUT we need to check in original eqⁿ to see if they are admissable

for $x=5$: $(x+4)>0$ + $(6-x)>0$ ✓

$x=-3$: $(x+4)>0$ + $(6-x)>0$ ✓

$\therefore$ Solution is $x = -3, 5$ (both worked out ☺)

B ⑨ $a^{\log_a 8 + \log_a 2} = a^{\log_a(8\times 2)}$

$= a^{\log_a 16}$

$= 16$

why? b/c a^x & $\log_a x$ are inverse operations

OR proof... let $y = \log_a x$ then $x = a^y$ (exp' form)

then $a^{\log_a x} = a^y = x$

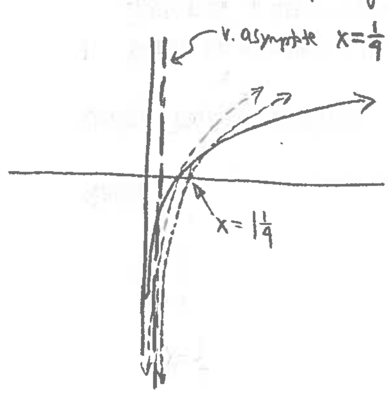
good rule to remember ☺ $\boxed{a^{\log_a x} = x}$

D ⑧ $y = \log_7 x \iff x = 7^y$ (note same bases — it helps!)

C ⑨ $y = \log_3(4x-1) + 3$ ⟶ or algebraically (easier):

or $y = \log\left(4\left(x-\frac{1}{4}\right)\right) + 3$

Can use graphing in this form:



You can see v. asymptote at $x=\frac{1}{4}$, so $x>\frac{1}{4}$

$y = \log(\underbrace{4x-1}_{\text{must be } >0 \text{ (positive)}}) + 3$

$\therefore 4x-1>0$

$4x>1$

$\therefore x>\frac{1}{4}$ is the domain

A ⑩ $\log_a\left(\frac{1}{a^b}\right) = \log_a\left(\frac{1}{a}\right)^b$

$= \log_a a^{-b}$

$= -b$

D ⑪ let $I(d)$ represent intensity of light at a depth of d m, where I_0 is the initial intensity of light.

Then $I(d) = I_0(0.98)^d$

note: • reduction of 2% is $100\% - 2\% = 1 - 0.02 = 0.98$ } base or decay factor

• k value is 1m b/c period is 1 m, which is why we have "d" instead of "d/k" in formula

Finding value of d when $I(d) = 10\%$ of I_0
$= 0.1 I_0$

$\therefore\ 0.1 I_0 = I_0(0.98)^d$ (divide LS+RS by I_0)

$0.1 = (0.98)^d$

log of both sides: $\log 0.1 = \log(0.98)^d$

$\log 0.1 = d\log(0.98)$ (exp rule)

$\therefore\ d = \frac{\log 0.1}{\log 0.98}$ (divide LS+RS by log 0.98)

$\doteq 114$ m

D ⑫ $5^{x-1} = 125^{3-x}$

$5^{x-1} = 5^{3(3-x)}$

$\therefore\ x-1 = 3(3-x)$

$x-1 = 9-3x$

$\therefore\ x = \frac{10}{4} = \frac{5}{2}$

C ⑬ pH of 7.6 → $10^{7.6}$
pH of 7 → 10^{7}

$\frac{10^{7.6}}{10^7} = 10^{7.6-7} = 10^{0.6} \doteq 3.98$

D ⑭ $a = b^c \iff \log_b a = c$

D ⑮ $\frac{10^{7.2}}{10^{5.8}} = 10^{7.2-5.8} = 10^{1.4} \doteq 25.12$

A ⑯ $\left(\frac{1}{4}\right)^{1-2x} = 8^{x-3}$

$\left(2^{-2}\right)^{(1-2x)} = \left(2^{3}\right)^{(x-3)}$ $\quad \because \ \frac{1}{4} = \left(\frac{1}{2}\right)^2 = 2^{-2}$

$2^{-2(1-2x)} = 2^{3(x-3)}$

like bases

$\therefore -2(1-2x) = 3(x-3)$

$-2 + 4x = 3x - 9$

$\therefore x = -7$

A ⑰ $y = \log_a x$ goes through $(1024, 5)$ $\Longrightarrow$ this eqⁿ is valid for $x = 1024$, $y = 5$:

$\therefore \ 5 = \log_a 1024$

$\therefore \ 1024 = a^5$ (exp form)

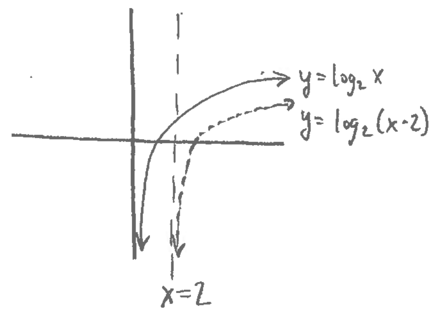
$\sqrt[5]{1024} = a$ (5th root of both sides)

$\therefore a = 4$

fun question ☺

B ⑱ $y = \log_2 \underbrace{(x-2)}$

graph of $y = \log_2 x$ moved 2 units right



C ⑲ $200\,(0.8)^5$ ← this happens 5 times ∵ goes through 5 filters

Initial amt.

20% removed

∴ 80% left ☺ ⓄⓇ think 100% − 20% = 80%

D ⑳ $\log_a x = 3$, $\log_a y = 4$

$\left(\log_a \frac{1}{xy}\right)^2 = \left[\log_a (xy)^{-1}\right]^2$

$= \left[-\log_a (xy)\right]^2$

$= (-1)^2 \left(\log_a x + \log_a y\right)^2$

$= (1)(3+4)^2$

$= 7^2$

$= 49$

OR $\left(\log_a \frac{1}{xy}\right)^2 = \left[\log_a 1 - \left(\log_a x + \log_a y\right)\right]^2$

$= \left[0 - (3+4)\right]^2$

$= (-7)^2$

$= 49$

note $\log_a 1 = \log_a a^0 = 0$

$\because a^0 = 1$ for any value "a"

B (21) $\log_4 C = x \iff C = 4^x$

A (22) $y = 2\log_4 \underbrace{(x-1)}_{>0} + 5$ $\qquad \therefore$ domain is $x-1>0$ or $x>1$

D (23) $25^{x+3} = 125^{2x-1}$

$5^{2(x+3)} = 5^{3(2x-1)} \implies 2(x+3) = 3(2x-1) \implies x = \frac{9}{4}$

$2x+6 = 6x-3$
$4x = 9$
$x = \frac{9}{4}$

D (24) $\log_4(x^2+1) - \log_4 6 = \log_4 5$

$\therefore \log_4\left[\frac{(x^2+1)}{6}\right] = \log_4 5$ $\qquad (\log_c a - \log_c b = \log_c \frac{a}{b})$

$\therefore \frac{(x^2+1)}{6} = 5$ $\qquad$ (same base on both sides)

$x^2+1 = 30$

$x^2 = 29$

$\therefore x = \pm\sqrt{29} \longrightarrow$ note $(x^2+1)>0$ for both $x=\sqrt{29}$ & $x=-\sqrt{29}$ $\quad \therefore x = \pm\sqrt{29}$

B (25) $y = \log_2(x+4)+1$ $\quad$ Find x-int.

occurs when $y=0 \implies 0 = \log_2(x+4)+1$

$\implies \log_2(x+4) = -1$

$\implies (x+4) = 2^{-1}$ $\qquad$ (exp form!) $\qquad \log_c a = b \iff a = c^b$

$\implies x+4 = \frac{1}{2}$

$\implies x = -3.5$

B (26) 6% /annum compounded monthly $\implies$ use $100\% + \frac{6\%}{12} = 1 + \frac{0.06}{12} = 1.005$ as base,

and this compounds 12 x per year $\implies$ $12t$ as exponent

So... $5000(1.005)^{12t}$

A (27) $\log(m^2n)^3 = \log(m^6n^3)$

$= \log m^6 + \log n^3$

$= 6\log m + 3\log n$

D (28) $\log_{5.3} 210 = \dfrac{\log 210}{\log 5.3} \doteq 3.21$

again... use $\log_a b = \dfrac{\log_c b}{\log_c a}$

*choose $c=10$ to use $\boxed{\log}$ on calc.

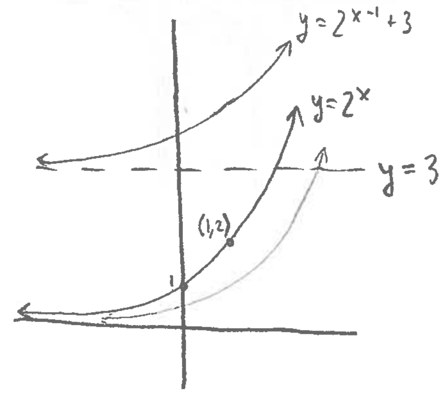
D (29) $27^{x+2} = \left(\frac{1}{3}\right)^{3-6x}$

$3^{3(x+2)} = 3^{-1(3-6x)}$

$\therefore 3x+6 = -3+6x$

$x = 3$

C (30) $f(x) = 2^{x-1}+3$ — affects y, affects x



f(x) represents graph of $y = 2^x$ moved 1 unit right & 3 units up

$y = 2^x$ has horizontal asymptote @ $y = 0$

$\therefore f(x)$ ——— // ——— $y = 3$

D (31) $\dfrac{10^{13}}{10^8} = 10^{13-8} = 10^5 = 100\,000$

A (32) $\log_3 (m+n) = 2$, $m+n > 0$

$\therefore (m+n) = 3^2$ (exp form)

$m+n = 9$

$\therefore m = 9-n$

A (33) $B = \dfrac{A}{C^2} \Longrightarrow \log B = \log \dfrac{A}{C^2}$

$= \log A - \log C^2$

$= \log A - 2\log C$

A (34) $N = 20g$
$C = 50g$
$t = 10$

$\therefore N = Ce^{kt}$

$20 = 50e^{k(10)}$

$\frac{2}{5} = e^{10k}$

$\therefore \ln \frac{2}{5} = \ln e^{10k}$

$\ln \frac{2}{5} = 10k \underbrace{\ln e}_{1}$

$\therefore k = \dfrac{\ln \frac{2}{5}}{10}$ or about -0.916

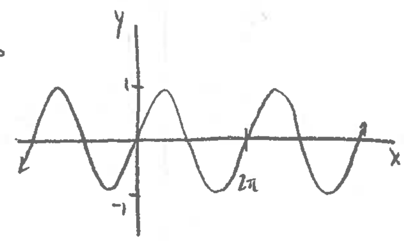
C 35) $\log y = \log(\sin x) \Rightarrow y = \sin x$ & note that the Cartesian plane is in terms of x & y,

So it's simply the graph of $y = \sin x$

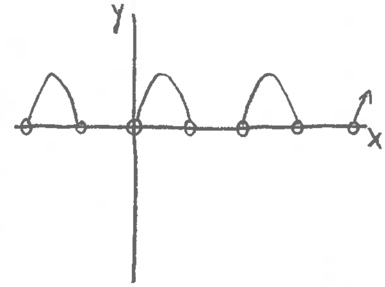
BUT $\log y \Rightarrow y > 0$ & $\log(\sin x) \Rightarrow x \neq k\pi$, for any integer k

not on test

So



becomes:



great question! (preview of next unit!)

C 36) $\log P - \log Q = \log \frac{P}{Q}$

B 37) $y = 2\log_3(x+4) - 5$

Important: logarithms have vertical asymptotes (i.e. $x = a$ for some a)
exponentials have horizontal asymptotes (i.e. $y = b$ for some b)

using our math ideas

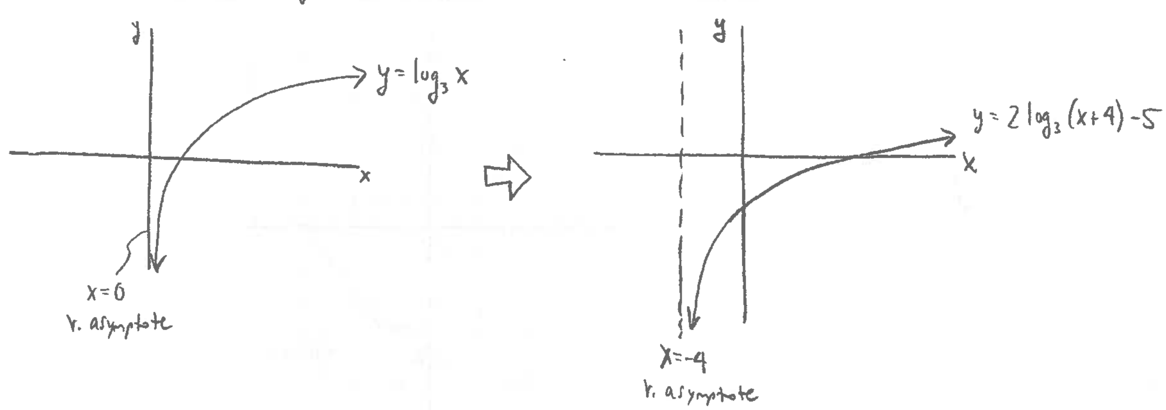
Since this is a log $\Longrightarrow$ v. asymptote

Find this from restriction on x-value

$y = 2\log_3 \underbrace{(x+4)} - 5$

$x > -4$ $\therefore$ v. asymptote at $x = -4$

OR you can graph $y = 2\log_3(x+4) - 5$ and see $x = -4$ is a v. asymptote.
This is more clear with your knowledge from Unit 1: Transformations



C 38) Solve $\log_5(x-3) = 2$

$(x-3) = 5^2$ (exp form)

$x - 3 = 25$

$x = 28$

B (39) $p = 100e^{-0.139a}$, given: $a = 5$ km

$\therefore p = 100e^{-0.139(5)}$

$\doteq 50$ kPa

A (40) $(\sqrt{a})^{6x-2} = (a^2)^{2x+3}$

$\therefore a^{\frac{1}{2}(6x-2)} = a^{2(2x+3)}$

$\therefore \frac{1}{2}(6x-2) = 2(2x+3)$

$3x - 1 = 4x + 6$

$\therefore x = -7$

D (41) $\frac{1}{2}$ life $\Rightarrow$ decay factor $\frac{1}{2}$

Initial amount is 100g

Final amount is 15g

period is 13 days

So... $15 = 100\left(\frac{1}{2}\right)^{t/13}$

$0.15 = (0.5)^{t/13}$

$\log 0.15 = \frac{t}{13}\log 0.5$

$\therefore t = \frac{13\log 0.15}{\log 0.5}$ or about 35.58 d

B (42) $\log c = 3$ So $\log 10c^2 = \log 10 + \log c^2$

$= 1 + 2\log c$

$= 1 + 2(3)$ $\because \log c = 3$

$= 7$

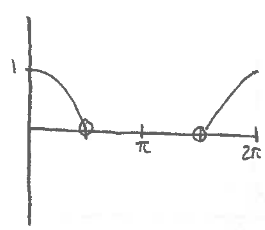
B (43) $\log \underbrace{(\cos x)}_{>0}$ $\therefore \cos x > 0 \Rightarrow 0 \le x < \frac{\pi}{2}$, $\frac{3\pi}{2} < x \le 2\pi$

(quad 1) (quad 4)

Not on test ☺

x in standard position,

So consider $0 \le x \le 2\pi$



B (44) $\log_k l = m \Leftrightarrow l = k^m$

A (45) $y = \log \underbrace{(2x+3)}_{>0}$ $\therefore$ domain is for $2x+3>0$ or $x > \frac{-3}{2}$

to be defined

D (46) $\frac{10^{8.5}}{10^{6.3}} = 10^{8.5-6.3} = 10^{2.2}$

B (47) $\log_3(x-6) + \log_3 x = 3$

restrictions: $x-6>0 \Rightarrow x>6$
(b/c can't take log of a number ≤ 0)
$x>0$ subsumed in ⑤

$$\log_3[(x-6)(x)] = \log_3 3^{3}$$

$$(x-6)(x) = 3^3$$

think 2 factors of -27 that sum to -6 and multiply to -27

$$x^2-6x-27=0$$

$$(x-9)(x+3)=0$$

$\therefore x=+9$, $\cancel{x=-3}$ inadmissable $\because x>6$

$\therefore$ $x=9$ is the only solution

D (48) $81^{x-1} = \left(\frac{1}{27}\right)^{x-4}$

$$3^{4(x-1)} = 3^{-3(x-4)}$$

$$\therefore 4(x-1) = -3(x-4)$$

$$4x-4 = -3x+12$$

$$x = \frac{16}{7}$$

C (49) $ab^x = c \Longrightarrow \log(ab^x) = \log c$

$$\log a + x\log b = \log c$$

(use multiplication (aka addition) + exponent rules)

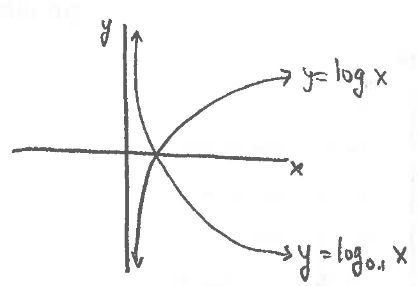
$$\therefore x = \frac{\log c - \log a}{\log b}$$

C (50) $0<a<1$ for $y=\log_a x$

Select a value for a, say $a=\frac{1}{10}$. Then $y = \log_{\frac{1}{10}} x = \dfrac{\log x}{\log(\frac{1}{10})}$

hmmm... what formula gave me this?!

Make it concrete. Easier ☺



$\therefore y = -\log x$

reflect graph of $y=\log x$ in x-axis ☺

or graph on calc if you forget

Written (1) Let $A(t)$ be the amount of Strontium-90 left after t days, measured in grams, with A_0 being the initial amount, in grams.

$\frac{1}{2}$-life $\Rightarrow$ decay factor is $\frac{1}{2}$

$A_0 = 200\,g$

$A(t) = 8g$

$k=28$ (period of half-life)

$$\therefore A(t) = A_0\left(\tfrac{1}{2}\right)^{t/k}$$

$$8 = 200\left(\tfrac{1}{2}\right)^{t/28}$$

$$0.04 = (0.5)^{t/28}$$

$\frac{8}{200} = 0.04$

$$\therefore \log 0.04 = \log(0.5)^{t/28}$$

$$\log(0.04) = \tfrac{t}{28}\log(0.5)$$

$$\therefore t = \frac{28\log 0.04}{\log 0.5}$$

$$\doteq 130 \text{ days}$$

② $\log_2 (2-2x) + \log_2 (1-x) = 5$

$\log_2 [(2-2x)(1-x)] = \log_2 2^5$

$\therefore (2-2x)(1-x) = 2^5$

$2-2x-2x+2x^2 = 32$

$2x^2 - 4x - 30 = 0$

$x^2 - 2x - 15 = 0$ (factor out 2 & divide from both sides)

factor

$(x-5)(x+3) = 0$

$\therefore \cancel{x=5}, x=-3$

inadmissable (see restrictions)

OR use quadratic eqⁿ:

$ax^2 + bx + c = 0$

$\Rightarrow x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$a=1$
$b=-2$
$c=-15$

$\Rightarrow x = 5, -3$

Restrictions: $(2-2x) > 0 \Rightarrow x < 1$

$(1-x) > 0 \Rightarrow x < 1$

(just coincidence that they're the same)

$\therefore x=-3$ is the only solution

③ $P = P_0 e^{kt}$

$k = ?$

$P = 1500$

$P_0 = 500$

$t = 8$

$\therefore 1500 = 500 e^{k(8)}$

$3 = e^{8k}$

$\ln 3 = \ln e^{8k}$

$\ln 3 = 8k (\ln e)$ equals 1

$\therefore k = \frac{\ln 3}{8}$ or about 0.137

Can we take the common logarithm (base 10) of both sides to find the solution?

TRY ☺

④ $\log_2 x + \log_2 (x-7) = 3$

$\log_2 x(x-7) = \log_2 2^3$

$\therefore x(x-7) = 2^3$

$x^2 - 7x - 8 = 0$

$(x-8)(x+1) = 0$

$\therefore x = 8, -1 \xrightarrow[\text{see restriction}]{} x = 8$ is the only solution

restrictions: $\underbrace{x > 0 \quad \& \quad x-7>0,\ x>7}_{\therefore\, x > 7}$

⑤ Let $A(t)$, in grams, be the amount of the radioactive substance after t years.
Let A_0 be the initial amount, in grams, and k is the half-life (period of decay factor $\frac{1}{2}$).

Then $A(30) = 150g$

$A_0 = 250g$

$t = 30$

$k = ?$

$A(t) = A_0 \left(\frac{1}{2}\right)^{t/k}$

$150 = 250 (0.5)^{30/k}$

$0.6 = (0.5)^{30/k}$

$\log 0.6 = \frac{30}{k} \log 0.5$

$k \doteq 40.71$ years

Could you take the natural logarithm (ln) of both sides instead?

TRY ☺

⑥ $2\log(3-x) = \log 4 + \log(6-x)$ restrictions: $\left.\begin{array}{l}(3-x)>0 \Rightarrow x<3 \\ (6-x)>0 \Rightarrow x<6\end{array}\right\} \Rightarrow x<3$

$\log(3-x)^2 = \log 4(6-x)$

$\left.\begin{array}{l}\therefore (3-x)^2 = 4(6-x) \\ 9-6x+x^2 = 24-4x\end{array}\right\}$ explanation: $(3-x)^2 = (3-x)(3-x)$

F – first
O – outer
I – inner
L – last

OR

	3	$-x$
3	9	$-3x$
$-x$	$-3x$	$+x^2$

$9-3x-3x+x^2$
$= x^2-6x+9$

$x^2-2x-15=0$

$(x-5)(x+3)=0$

$\therefore \cancel{x=5}$, $x=-3$ ∴ $x=-3$ is the only solution

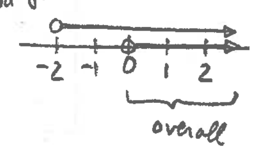
inadmissable (check restrictions)

⑦ $\log_2 x = 3 - \log_2(x+2)$ ⟹ restrictions: $x>0$

$x+2>0 \Rightarrow x>-2$

$\therefore x>0$ overall

visually:



$\log_2 x = \log_2 2^3 - \log_2(x+2)$

$\log_2 x = \log_2\left(\frac{8}{x+2}\right)$

$\therefore x = \frac{8}{x+2}$

$x(x+2) = 8$ ($\cdot x$ both sides by $x+2$)

$x^2+2x-8=0$

$(x+4)(x-2)=0 \Longrightarrow \cancel{x=-4}$, $x=2$ ∴ $x=2$ is the solution

inadmissable

⑧ let $A(t)$ mg be the amount of a radioactive substance after t weeks, where A_0 is the initial amount, and k is the half-life in weeks.

$A(73) = 450$ mg
$A_0 = 3150$ mg
$t = 73$ wks
$k = ?$

$A(t) = A_0\left(\frac{1}{2}\right)^{t/k}$

$450 = 3150\left(\frac{1}{2}\right)^{\frac{73}{k}}$

$\frac{1}{7} = \left(\frac{1}{2}\right)^{73/k}$ (divide both sides by 3150 + reduce fraction)

$\therefore \ln\left(\frac{1}{7}\right) = \ln\left(\frac{1}{2}\right)^{73/k}$ ⟵ can take log of any base of both sides ... w/ calc we usually use $\log_{10}$ or $\log_e$ — choose ($\boxed{\text{LOG}}$) ($\boxed{\text{LN}}$) whichever you feel like ☺

$\ln\left(\frac{1}{7}\right) = \frac{73}{k}\ln\left(\frac{1}{2}\right)$

$k = \frac{73\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{1}{7}\right)}$

$\doteq 26$ weeks