Day 15 | Test: Functions

Review: ① Determine the equation of a circle with radius 3 units, centred at the origin. (Apply)

② a) Identify the points associated with $P(\theta)$, where θ is an integral multiple of $\frac{\pi}{4}$ so that $0 \le \theta \le 2\pi$. Draw the points on the unit circle.

b) Identify the points associated with $P(\theta)$, where θ is an integral multiple of $\frac{\pi}{6}$ so that $0 \le \theta \le 2\pi$. Draw the points on the unit circle.

Hint: Special triangles!

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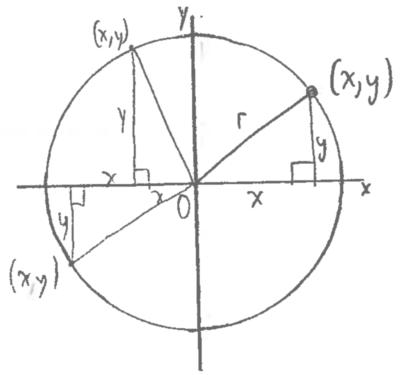
① Think of where the equation for the unit circle came from (see diagram). We can generalize for any radius r, as long as circle is centred at origin.



for unit circle $r=1$

Notice wherever pt (x,y) is, the same relationship holds true: Pythagorean relationship between x, y, r:

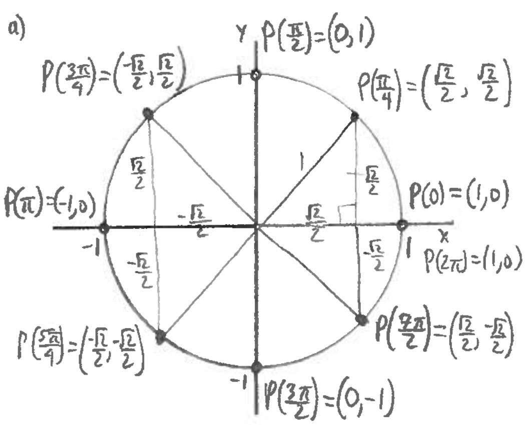
$$\boxed{x^2 + y^2 = r^2}$$ — general eqn of circle centred at origin

for $r=3$...

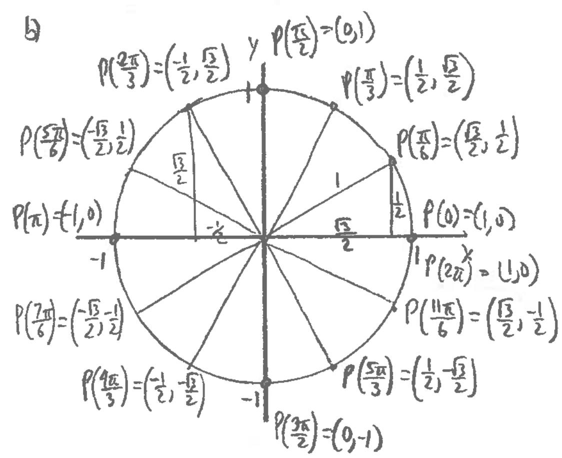
$$\boxed{x^2 + y^2 = 9}$$

② Use special triangles and the patterns you see to find all the pts $P(\theta) = (x,y)$...

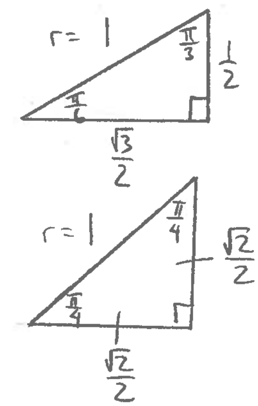
a)



b)

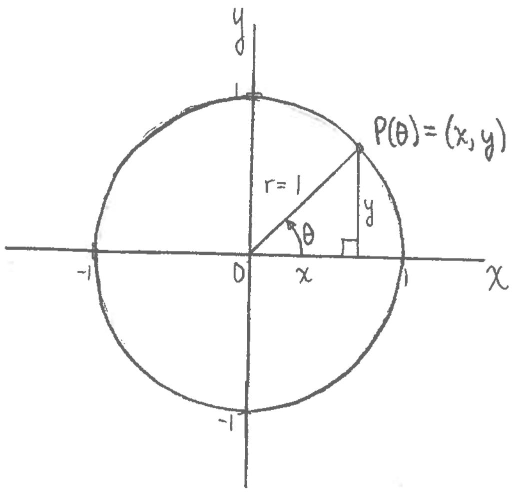


for unit circle:



* What patterns do you see to help get these points?

4.3 Trigonometric Ratios



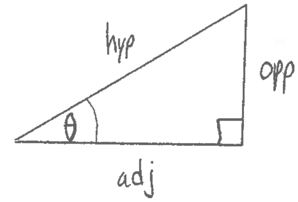
$$\boxed{P(\theta) = (\cos\theta, \sin\theta)}$$

Recall: reference angle; by "opposite" we mean length of opposite side

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$



Now if $P(\theta) = (x, y)$ is a point on the unit circle (see diagram, left), then:

$$\cos\theta = \frac{x}{r} = \frac{x}{1} = x \implies x = \cos\theta$$

$$\sin\theta = \frac{y}{r} = \frac{y}{1} = y \implies y = \sin\theta$$

So that $P(\theta) = (x, y) = (\cos\theta, \sin\theta)$

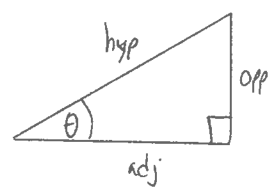
Reciprocal Trig Ratios

Cosecant, secant, and cotangent are the reciprocals of sine, cosine, and tangent, respectively:

"cosecant": $\csc\theta = \dfrac{1}{\sin\theta}$ ie. $\csc\theta = \dfrac{\text{hyp}}{\text{opp}}$

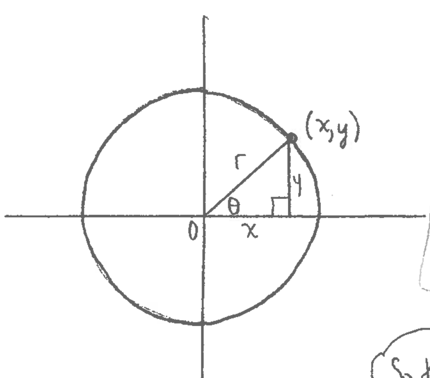
"secant": $\sec\theta = \dfrac{1}{\cos\theta}$ ie. $\sec\theta = \dfrac{\text{hyp}}{\text{adj}}$

"cotangent": $\cot\theta = \dfrac{1}{\tan\theta}$ ie $\tan\theta = \dfrac{\text{adj}}{\text{opp}}$



Summary:

For any circle centred at origin:



$\sin\theta = \dfrac{y}{r}$	$\csc\theta = \dfrac{r}{y}$
$\cos\theta = \dfrac{x}{r}$	$\sec\theta = \dfrac{r}{x}$
$\tan\theta = \dfrac{y}{x}$	$\cot\theta = \dfrac{x}{y}$

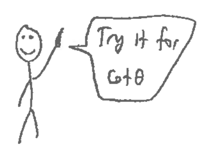
Notice: $\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}}$ + $\boxed{\cot\theta = \frac{\cos\theta}{\sin\theta}}$

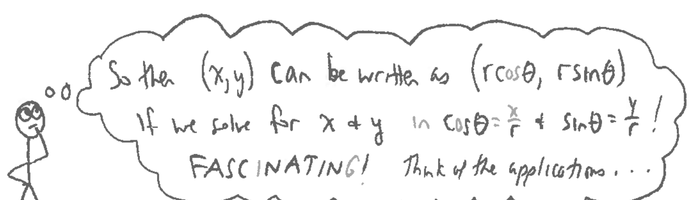
proof: $\dfrac{\sin\theta}{\cos\theta} = \dfrac{\frac{y}{r}}{\frac{x}{r}}$

$= \left(\dfrac{y}{r}\right)\left(\dfrac{r}{x}\right)$

$= \dfrac{y}{x}$

$= \tan\theta$





Having difficulty remembering the 3 primary trig ratios? Use the acronym: SOH CAH TOA!

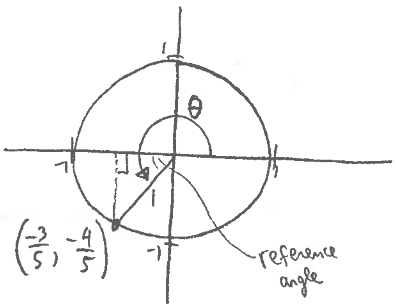
SOH CAH TOA

$\sin\theta = \frac{O}{H}$ $\quad \cos\theta = \frac{A}{H}$ $\quad \tan\theta = \frac{O}{A}$

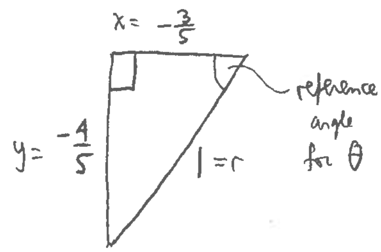
They are all found on your formula sheet — go check it out!! :)

Ex 1 Determine the 6 trig ratios for a pt $\left(-\frac{3}{5}, -\frac{4}{5}\right)$ on the unit circle.

Soln: get into the habit of sketching a diagram!



Look at triangle up close:



$\sin\theta = \frac{y}{r} = \frac{\frac{-4}{5}}{1} = \frac{-4}{5}$

$\cos\theta = \frac{x}{r} = \frac{\frac{-3}{5}}{1} = \frac{-3}{5}$

$\tan\theta = \frac{y}{x} = \frac{\frac{-4}{5}}{\frac{-3}{5}} = \frac{-4}{-3} = \frac{4}{3}$

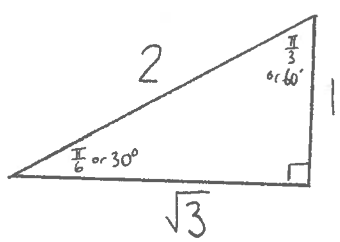
$\underbrace{\csc\theta = \frac{-5}{4} \quad \sec\theta = \frac{-5}{3} \quad \cot = \frac{3}{4}}_{\text{Just reciprocals of above :)}}$

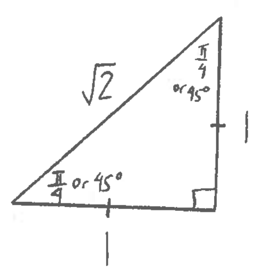
think: $\frac{-4}{5} \div \frac{-3}{5} = -4 \div -3 = \frac{-4}{-3} = \frac{4}{3}$

same!

Recall Special Triangles for exact values:

Check last day's notes (4.2). Here's another way to remember them (easier ratios to remember :))



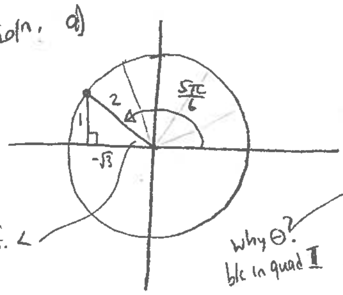


Using these triangles, we can determine exact values for trig ratios of angles: $\frac{\pi}{6}$ or 30°, $\frac{\pi}{4}$ or 45°, $\frac{\pi}{3}$ or 60°

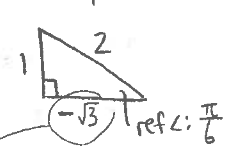
and their reference angles :)

Ex 2 Determine exact value of: a) $\cos\left(\frac{5\pi}{6}\right)$ b) $\cot(-45°)$

Soln: a)

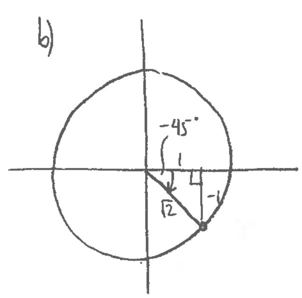


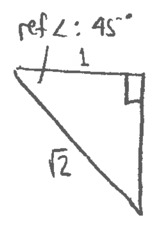
Close up:



why ⊖? b/c in quad II $\therefore \cos\frac{5\pi}{6} = \frac{-\sqrt{3}}{2}$

b)





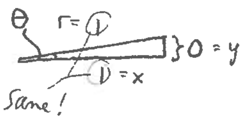
$\cot(-45°) = \frac{1}{-1} = -1$

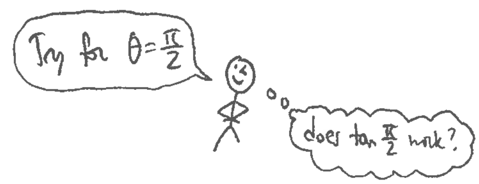
Angles of 0 + $\frac{\pi}{2}$ (and then multiples are also trivial when examining a triangle)

ex $\sin 0 = \frac{y}{r} = \frac{0}{1} = 0$

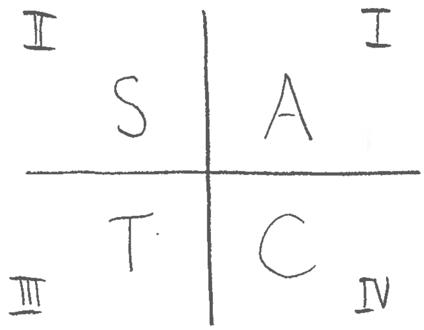
$\cos 0 = \frac{x}{r} = \frac{1}{1} = 1$

$\tan 0 = \frac{y}{x} = \frac{0}{1} = 0$





So why are some trig ratios ⊕ and others ⊖ ?



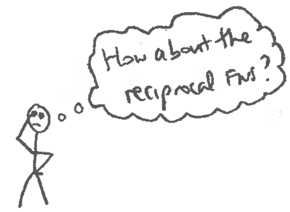
Use CAST rule (or ASTC ... "All Students Take Calculus")

C → only cosine ⊕ in quad IV

A → ALL trig ratios ⊕ in quad I

S → only sine ⊕ in quad II

T → only tangent ⊕ in quad III



Approximate values

Use calculator + check to see if you're in RAD or DEG mode.

* Some calculators don't give values in correct quadrant, so use CAST rule to check sign.

Ex: $\cos 260° \approx -0.1736$, $\csc(-70°) = \frac{1}{\sin(-70°)} \approx -1.0642$, $\sin 4.2 \approx -0.8716$

radian measure!

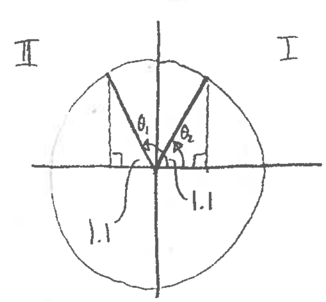
To find an angle when given a ratio, you need to use the inverse trig functions on your calculator:

Ex 3 Determine measures of all angles satisfying the following, in given domain.

a) $\sin\theta = 0.879$, $0 \le \theta < 2\pi$

b) $\sec\theta = \frac{-2}{\sqrt{3}}$, $-2\pi \le \theta < 2\pi$ (give exact answer)

Sol'n: a) by CAST $\sin\theta > 0 \Rightarrow$ quad I or II



$\sin^{-1} 0.879 \approx 1.1$

So 1 angle is 1.1

Other one is $\pi - 1.1$

or about 2.0

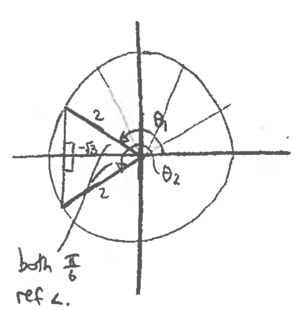
$\therefore \theta \approx 1.1, 2.0$

θ_2 θ_1

Notice the symmetry!

b) by CAST $\sec\theta < 0 \Rightarrow$ quad II or III

reciprocal of cosine (can think $\sec\theta = \frac{-2}{\sqrt{3}} \Rightarrow \cos\theta = \frac{-\sqrt{3}}{2}$ if easier)



In any case, $\theta = \frac{\pi}{6}$ (special Δ's) is the reference ∠ in quad II or III

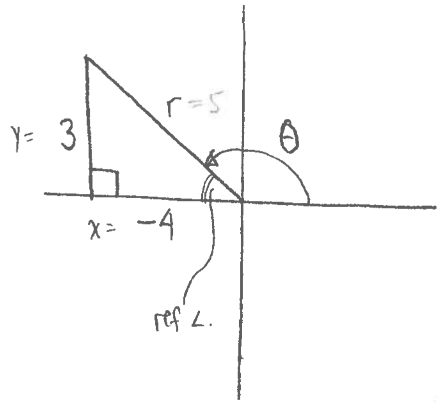
Can see $\theta_1 = \frac{5\pi}{6}$, $\theta_2 = \frac{7\pi}{6}$

But domain $-2\pi \le \theta < 2\pi$, so we also need to subtract 2π from each to get 2 more solutions! ex $\frac{7\pi}{6} - \frac{12\pi}{6} = -\frac{5\pi}{6}$

$\frac{5\pi}{6} - \frac{12\pi}{6} = -\frac{7\pi}{6}$

$\therefore \theta = \frac{-7\pi}{6}, \frac{-5\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$

Ex 4 The pt $(-4,3)$ lies on the terminal arm of an angle θ in standard position.
Find the exact values of the 6 trig ratios for θ.



notice Pythagorean relationship:

$$(3)^2 + (-4)^2 = r^2$$
$$9 + 16 = r^2$$
$$25 = r^2$$
$$\therefore r = 5$$

(why must it be ⊕ ?)

use reference ∠ :

$\sin\theta = \frac{y}{r} = \frac{3}{5} \Rightarrow \csc\theta = \frac{5}{3}$

$\cos\theta = \frac{x}{r} = \frac{-4}{5} \Rightarrow \sec\theta = -\frac{5}{4}$

$\tan\theta = \frac{y}{x} = \frac{3}{-4} \Rightarrow \cot\theta = -\frac{4}{3}$

HWK pg 201 – 203 # (1-6) a, d ; 8, 10, 12, 13