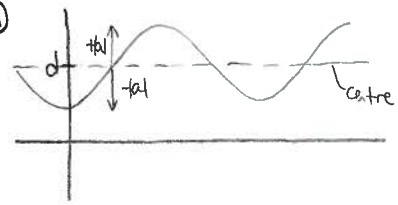
Day 18 Review

① What is the range of $y = a\cos b(x-c) + d$ for any $a, b, c, d \in \mathbb{R}$, $b \neq 0$?

② Sketch a graph of $y = -2\sin(\pi x + \pi)$. Describe the transformations necessary to obtain the graph of this function from the graph of $y = \sin x$.

Sol'n: ①



min value: $d - |a|$

max value: $d + |a|$

$\therefore$ range is $d - |a| \leq y \leq d + |a|$

or $\{y \mid d - |a| \leq y \leq d + |a|,\ y \in \mathbb{R}\}$

② Look at FN to see transformations from $y = \sin x$:

$$y = -2\sin(\pi x + \pi) \longrightarrow y = -2\sin \pi(x+1)$$

- $-$: reflection in x-axis
- 2: vertical expansion by factor 2 wrt x-axis
- π: horizontal compression by factor $\frac{1}{\pi}$ wrt y-axis

multiplicative transformations first

- $+1$: horizontal translation (phase shift) by 1 unit left

additive transformations last

Also worth noting:

$a = -2 \Rightarrow$ amplitude $= |-2| = 2$

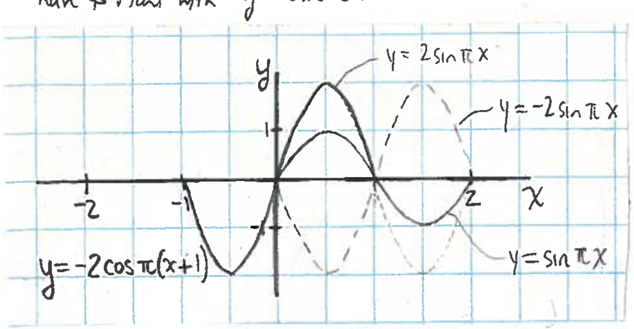
period $= \dfrac{2\pi}{|b|} = \dfrac{2\pi}{\pi} = 2$

*can also get period by performing horizontal compression of factor $\frac{1}{\pi}$ on original period of 2π to get 2

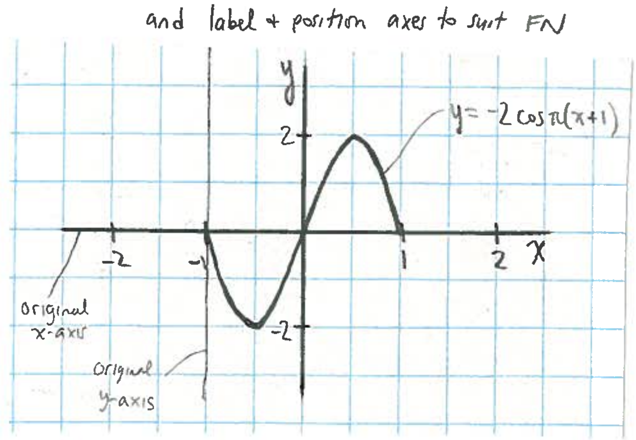
Sketch:

method 1 — begin with base FN $y = \sin x$ & perform above transformations

This works well when the period is a rational multiple of π, but in this case the period is 2 ... so we would at least have to start with $y = \sin \pi x$



method 2 — draw the basic shape of a sinusoidal curve and label + position axes to suit FN



* the graph of FN you see above used to be $y = -\sin x$, but then we scaled x & y axes to fit FN and moved the original y-axis to accommodate the phase shift. A more efficient method!!

5.3 – The Tangent Function

Recall: $\sin\theta = \frac{y}{r}$ + $\cos\theta = \frac{x}{r}$, so we can write $\tan\theta = \frac{y}{x}$ in terms of $\sin\theta$ + $\cos\theta$:

$$\left.\begin{array}{l} \sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta \\ \cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta \end{array}\right\} \tan\theta = \frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \frac{\sin\theta}{\cos\theta}, \text{ if } r \neq 0$$

$\therefore$ $\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}}$ This also means that $\boxed{\cot\theta = \frac{\cos\theta}{\sin\theta}}$

We'll use these identities next chapter

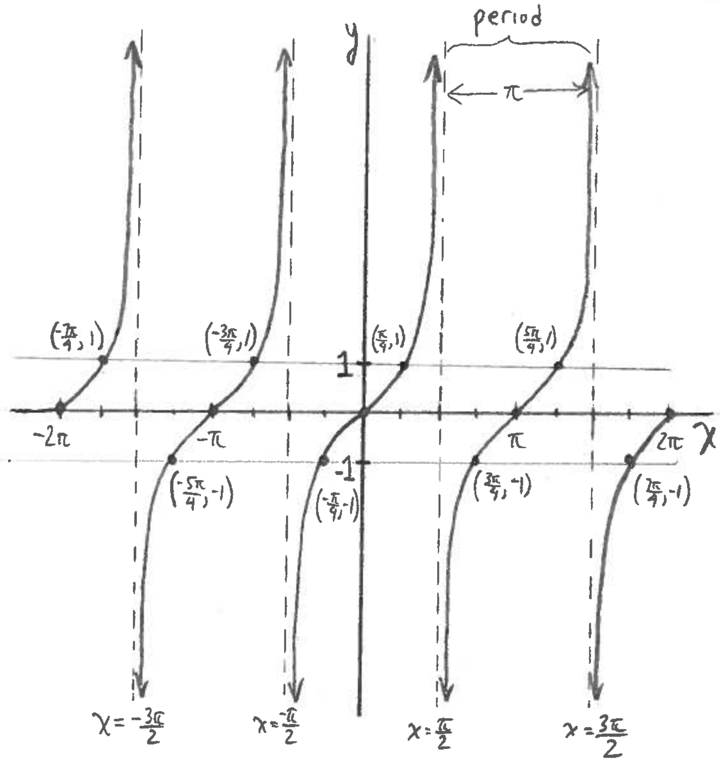
Ex 1 Graph $y = \tan x$ on your graphing calculator.

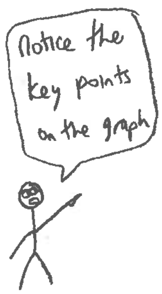
a) Sketch the graph in your notebook and label asymptotes + key points on $[-2\pi, 2\pi]$

b) Determine the domain, range, x-intercepts, y-intercepts, and period for $y = \tan x$

c) Write a general equation for the asymptotes of $y = \tan x$

Sol'n a)





b) domain is:

$\{x \mid x \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}, x \in \mathbb{R}\}$

range is:

$\{y \mid y \in \mathbb{R}\}$

no min + no max

x-intercepts occur at:

$x = k\pi, k \in \mathbb{Z}$

y-intercept is 0.

period is π.

c) Get general equation for the vertical asymptotes from the domain:

$$x = \frac{\pi}{2} + k\pi, \text{ where } k \text{ is any integer (i.e., } k \in \mathbb{Z})$$

Ex 2 A small plane is flying at a constant altitude of 5000m directly toward an observer standing on level ground.

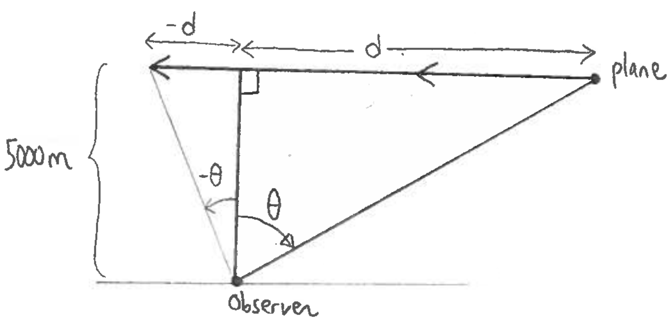
a) Determine an equation relating the horizontal distance, d, in metres, from the observer to the plane and the angle, θ, in degrees, formed from the vertical to the plane.

b) Sketch the graph of the function. What is an appropriate domain?

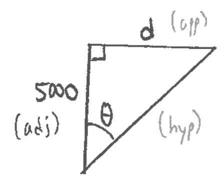
c) What are the asymptotes and what do they represent?

d) What is occuring when $\theta = 0°$

Sol^n: It helps to draw a picture:



a) From the triangle, we can see tangent relates d, θ, + 5000:

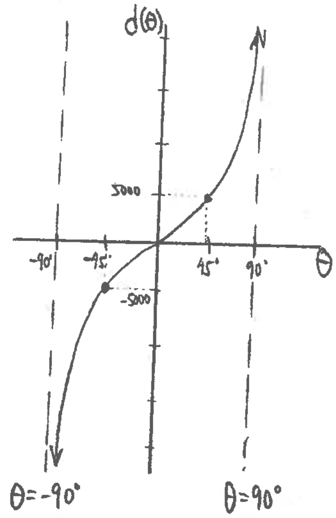


$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{d}{5000}$$

$$\therefore\ d = 5000\tan\theta$$

Can write as Fn in θ: $\boxed{d(\theta) = 5000\tan\theta}$

b) Use your graphing calculator to help sketch. Notice that an appropriate domain would be $\theta \in (-90°, 90°)$, since $\theta \to \pm 90°$ would mean the plane is infinitely far away! Using $d \in [-10000, 10000]$, we get:



c) asymptotes represent the plane at an "infinite" distance away. $\theta = 90°$ represents the plane an infinite distance in front of observer and $\theta = -90°$ is the plane an infinite distance behind observer.

d) When $\theta = 0°$, the plane is directly overhead.

HWK: p 263-264 #6, 9, 10

5.4 Equations & Graphs of Trig Functions

Ex1 Determine the general solution for $16 = 6\cos\frac{\pi}{6}x + 14$. Express solutions to the nearest hundredth.

Sol'n graphically:

method I

set $Y1 = 16$

$Y2 = 6\cos((\pi/6)X) + 14$

use $\boxed{\text{INTERSECT}}$ FN

* we want a window that shows at least 1 full cycle

from Y_2, get period:

$$\text{Period} = \frac{2\pi}{(\pi/6)} = 12$$

helps w/ x-values in window

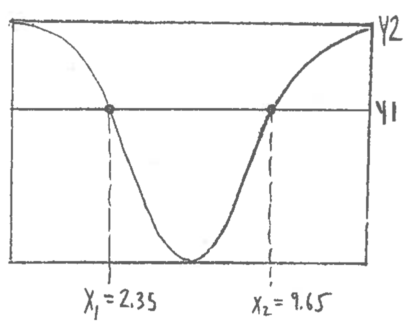
(think transformations: $\underbrace{2\pi}_{\text{original period}} \times \underbrace{\frac{6}{\pi}}_{\text{stretch factor}} = 12$)

and range: $\underbrace{14}_{d} \pm \underbrace{6}_{|a|} \Rightarrow [8, 20]$

helps w/ y-values in window

Possible Window:

	min	max
X	0	12
Y	8	20



Since the period is 12, the general solution is:

$$X_1 = 2.35 + 12k, \quad k \in \mathbb{Z}$$

$$X_2 = 9.65 + 12k, \quad k \in \mathbb{Z}$$

method II

Rearrange: $16 = 6\cos\frac{\pi}{6}X + 14$

$0 = 6\cos\frac{\pi}{6}X - 2$

set $Y1 = 6\cos((\pi/6)X) - 2$

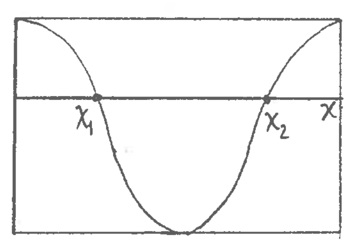
use $\boxed{\text{ZERO}}$ FN

* we want a window that shows at least 1 full cycle.

Obtain period of 12 as in method I, but different range: $\underbrace{-2}_{d} \pm \underbrace{6}_{|a|} \Rightarrow [-8, 4]$

Possible Window:

	min	max
X	0	12
Y	-8	4



$x_1 = 2.35$

$x_2 = 9.65$

As in method I, the general solution is:

$$X_1 = 2.35 + 12k, \quad k \in \mathbb{Z}$$

$$X_2 = 9.65 + 12k, \quad k \in \mathbb{Z}$$

With complicated FNs like:

ex: $\tan^2 x = \sin^2 x \cos x + x$

Graphing is the way to go!! :)

Continued Sol^n:

algebraically

Solve an easier problem... let $\theta = \frac{\pi}{6}x$ Then $16 = 6\cos\frac{\pi}{6}x + 14$ becomes:

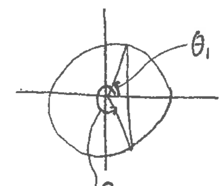
$$16 = 6\cos\theta + 14$$
$$-14 \qquad\qquad -14$$
$$\frac{2}{6} = \frac{6\cos\theta}{6}$$
$$\frac{1}{3} = \cos\theta$$

$\therefore\ \theta = \cos^{-1}\left(\frac{1}{3}\right) \approx 1.2309$

now $\theta > 0 \Rightarrow \theta$ could be in quad I or IV by CAST

$\therefore\ \theta_1 \approx 1.2309$

$\theta_2 \approx 2\pi - 1.2309 \approx 5.0522$



BUT $\theta = \frac{\pi}{6}x$ ... So $1.2309 \approx \frac{\pi}{6}x$ + $5.0522 \approx \frac{\pi}{6}x$

$\left(\frac{6}{\pi}\right)1.2309 \approx x$ $\qquad$ $\left(\frac{6}{\pi}\right)5.0522 \approx x$

$\boxed{2.3509 \approx x}$ $\qquad$ $\boxed{9.6490 \approx x}$

Now period of function is 12 radians (see previous page), so general solution is:

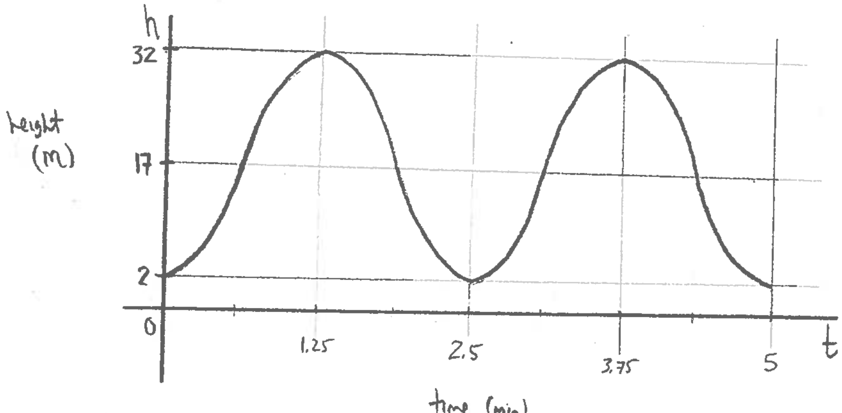
$$x \approx 2.35 + 12k\ ,\ k \in \mathbb{Z}$$
$$x \approx 9.65 + 12k\ ,\ k \in \mathbb{Z}$$

Ex 2 A Ferris wheel with radius 15m makes 2 rotations in 5 minutes. The centre is 17m from the ground. Passengers board at the Ferris wheel's lowest point.

a) Write an equation to model the height, h, in metres above the ground after time, t, minutes of a passenger once the ride begins.

b) How long will it take a passenger to first reach a height of 20m?

Sol'n: a) A graph of the situation will help to determine the equation:



A cosine graph (reflected in x-axis) appears to best fit the graph:

$$h = a\cos b(t-c) + d$$

notice $c=0$ because $a<0$ takes care of the need for a phase shift.

The centre gives: $d = 17$

We see amplitude is 15m, Combining with reflection in x-axis, we get $a = -15$

Period is 2.5m, so then

$$\frac{5}{2} = \frac{2\pi}{b} \Rightarrow b = \frac{4\pi}{5}$$

$$\therefore h = -15\cos\frac{4\pi}{5}t + 17$$

or with fn notation:

$$\boxed{h(t) = -15\cos\frac{4\pi}{5}t + 17}$$

* Check with graphing calculator ☺

b) Solve for t when $h(t) = 20$m:

$$20 = -15\cos\frac{4\pi}{5}t + 17$$

Use any of the 3 ways described for Ex 1

Easiest on graphing calculator:

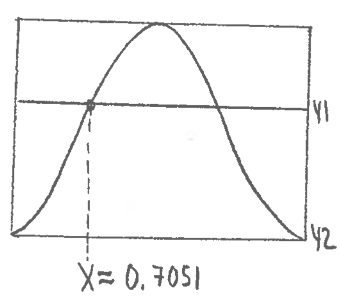
Y1 = 20

Y2 = −15 cos((4π/5)X) + 17

Use $\boxed{\text{INTERSECT}}$ Function

Possible Window:

	min	MAX
X	0	2.5
Y	2	32



$X \approx 0.7051$

$\therefore$ It takes the passenger about 0.7 min or about 42 s to reach a height of 20m the first time.

HWK: p275 – 280 : # 5, 9, 19–21