Day 19

Review

① Determine an infinite set of asymptotes for $y = \tan\left(\frac{x}{2} + \pi\right)$.

② Determine a cosine equation that has the following general solution:

$$\frac{\pi}{2} + n\pi, \quad \frac{\pi}{6} + 2n\pi, \quad \frac{11\pi}{6} + 2n\pi$$

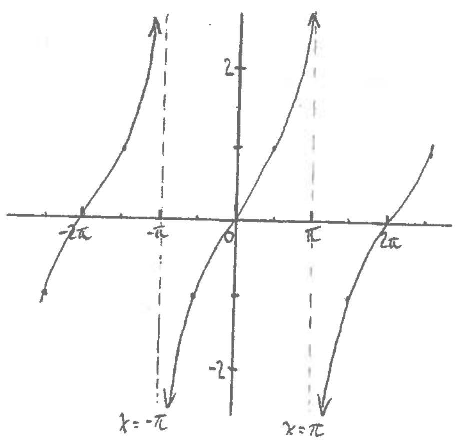
③ The function $h(t) = 3.9 \sin 0.16\pi(t-3) + 6.5$ gives the depth of water, h metres, at any time, t hours, during a certain day. A cruise ship needs at least 8 m of water to dock safely. Use the graph of the function to estimate the number of hours in a 12 h period beginning at $t=0$ during which the cruise ship can dock safely.

Soln: ① It's easiest just to graph $Y1 = \tan((X/2) + \pi)$ over an appropriate domain (the period is useful for this)

-OR- $y = \tan \frac{1}{2}(x + 2\pi)$

original period of $y = \tan x$ is π, so $\times 2$ gives new period 2π (or use period $= \frac{\pi}{b} = \frac{\pi}{1/2} = 2\pi$)

phase shift is 2π left — same as period! So has no affect on look of graph



If we didn't know the period before, we see it now in the graph: 2π

We can see that asymptotes are:

$$\{x = \pi + 2k\pi, k \in \mathbb{Z}\}$$

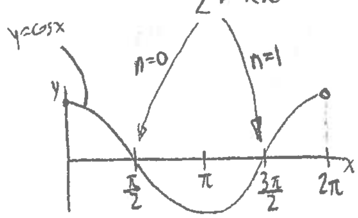
So this simplifies to $y = \tan\frac{x}{2}$ and apply horizontal stretch to original asymptotes + new period:

$y = \tan x$ asymptotes: $x = \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$

$y = \tan\frac{x}{2}$ asymptotes: $x = 2\left(\frac{\pi}{2} + k\pi\right) = \pi + 2k\pi, k \in \mathbb{Z}$

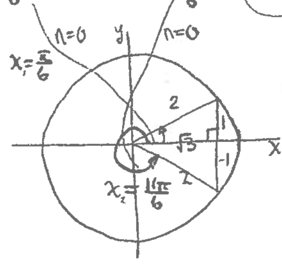
② Visualize the solutions in terms of $y = \cos x$ on $[0, 2\pi)$:

$\frac{\pi}{2} + n\pi$



$\therefore \cos x = 0$ for these values of x

$\frac{\pi}{6} + 2n\pi$ $\quad$ $\frac{11\pi}{6} + 2n\pi$ — this confirms $y = \cos x$ is an appropriate fit because period is 2π



In each case...

$\cos x = \frac{y}{r} = \frac{\sqrt{3}}{2}$

$\therefore \cos x - \frac{\sqrt{3}}{2} = 0$

or

$2\cos x - \sqrt{3} = 0$

For $\cos x$ or $(2\cos x - \sqrt{3})$ to be zero, then:

$$\cos x(2\cos x - \sqrt{3}) = 0$$

or $2\cos^2 x - \sqrt{3}\cos x = 0$

③ $Y1 = 3.9\sin(0.16\pi(X-3)) + 6.5$

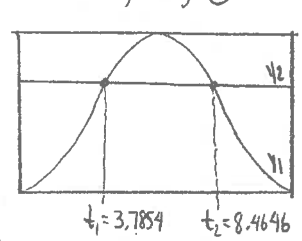
$Y2 = 8$

and use $\boxed{\text{INTERSECT}}$ fn

with appropriate window:

	min	max
X	0	12
Y	2.6 (6.5−3.9)	10.4 (6.5+3.9)

* any range including $y=8$ will do, really ☺



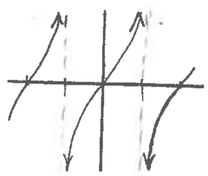
$\therefore$ Total hours in 12h interval is $t_2 - t_1$ or about $\boxed{4.7\text{ h}}$

Reciprocal, Quotient, and Pythagorean Identities (6.1 + applications in 6.3 + 6.4)

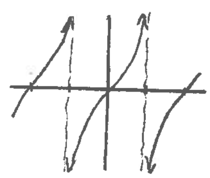
An identity is an equation that is true for all permissable values of the variable.

Ex $\tan x = \frac{\sin x}{\cos x}$ is a trig identity ; $\cos x = -\sin x$ is an equation

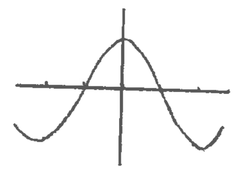
Compare graphs to verify this:



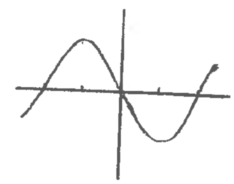
$y = \tan x$



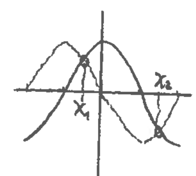
$y = \frac{\sin x}{\cos x}$



$y = \cos x$



$y = -\sin x$



But they are equal for $x = \frac{-\pi}{4}, \frac{3\pi}{4}$ on $[-\pi, \pi]$ ... that's "solving an equation" ☺

* So... you can verify a trig identity graphically if the left side (LS) and right side (RS) have the same graph ☺

You can also verify algebraically ... [coming soon]

Some identities you already know:

Reciprocal Identities: $\csc x = \frac{1}{\sin x}$ $\sec x = \frac{1}{\cos x}$ $\cot x = \frac{1}{\tan x}$

Quotient Identities: $\tan x = \frac{\sin x}{\cos x}$ $\cot x = \frac{\cos x}{\sin x}$

Ex1 Simplify the expression $\frac{\cot x}{\csc x \cos x}$ and state any non-permissable values.

Sol^n:
$$\frac{\cot x}{\csc x \cos x} = \frac{\frac{\cos x}{\sin x}}{\left(\frac{1}{\sin x}\right)\frac{\cos x}{1}}$$
$$= \frac{\cos x}{\sin x} \div \frac{\cos x}{\sin x}$$
$$= 1$$

non-permissable values:

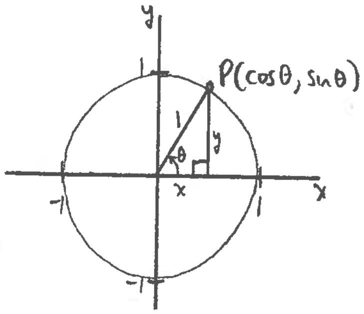
$$\left.\begin{aligned} \cot x &\Longrightarrow x \neq k\pi,\ k \in \mathbb{Z} \\ \csc x &\Longrightarrow x \neq k\pi,\ k \in \mathbb{Z} \end{aligned}\right\} \text{check with graphing calc!}$$

Also...
$\csc x \cos x \neq 0$ (denominator)

$\therefore$ $\csc x \neq 0$ (there is no x that makes $\csc x = 0$, so no restriction)
$\cos x = 0 \Longrightarrow x \neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$

All together, $x \neq \frac{k\pi}{2},\ k \in \mathbb{Z}$
(this includes both cases above)

Recall...



Pythagorus:

$$x^2 + y^2 = 1^2$$

$$(\cos\theta)^2 + (\sin\theta)^2 = 1$$

$$\boxed{\cos^2\theta + \sin^2\theta = 1}$$ ⟵ 1 of 3 Pythagorean identities!

* Divide each term in this identity by $\sin^2\theta$ OR $\cos^2\theta$ to get the 2 other Pythagorean Identities — TRY IT!!

Pythagorean Identities:

$$\cos^2\theta + \sin^2\theta = 1 \qquad \cot^2\theta + 1 = \csc^2\theta \qquad 1 + \tan^2\theta = \sec^2\theta$$

Ex 2 Suppose $\dfrac{\cos x}{1 - \sin x} = \dfrac{1 + \sin x}{\cos x}$

a) Show this is true for $x = \frac{\pi}{6}$ algebraically.

b) Show this is an identity graphically.

c) Prove this identity algebraically. State any restrictions.

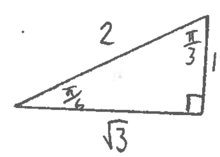
Soln: a) Show $x = \frac{\pi}{6}$ yields same solution on LS + RS of the equation.

$$LS = \frac{\cos\frac{\pi}{6}}{1 - \sin\frac{\pi}{6}} = \frac{\frac{\sqrt{3}}{2}}{1 - \frac{1}{2}} = \frac{\sqrt{3}}{2} \div \frac{1}{2} = \sqrt{3} \div 1 = \sqrt{3}$$

Same! ÷ numerators

$$RS = \frac{1 + \sin\left(\frac{\pi}{6}\right)}{\cos\left(\frac{\pi}{6}\right)} = \frac{1 + \frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{3}{2} \div \frac{\sqrt{3}}{2} = \frac{3}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\cancel{3}\sqrt{3}}{\cancel{3}} = \sqrt{3}$$

aside:

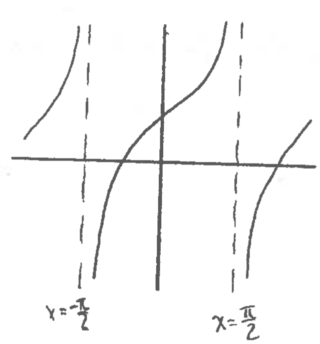


Use special triangle to get exact values

Since LS = RS, then statement is true for $x = \frac{\pi}{6}$

b) Set Y1 = COS(X)/(1 − SIN(X))

Y2 = (1 + SIN(X))/COS(X)



* You can do them separately + compare OR see that they overlap exactly ☺

You can also see restrictions from graph (to use in c))

c) method 1 ... begin with complicated side (if one) and use algebra + identities to get other side

$$LS = \frac{\cos x}{1-\sin x}$$

$$= \frac{\cos x}{(1-\sin x)} \times \frac{1+\sin x}{1+\sin x}$$ = 1 so doesn't change sol^n

$1-\sin x$ and $1+\sin x$ are called conjugates & multiply to a difference of squares!!

$$= \frac{\cos x(1+\sin x)}{(1-\sin x)(1+\sin x)}$$

$$= \frac{\cos x(1+\sin x)}{1^2-\sin^2 x}$$ difference of squares: $(a-b)(a+b) = a^2-b^2$

$$= \frac{\cos x(1+\sin x)}{\cos^2 x}$$ $\because \sin^2 x + \cos^2 x = 1$, $\cos^2 x = 1-\sin^2 x$

$$= \frac{1+\sin x}{\cos x}$$

$= RS$

$\because LS = RS$ $\therefore$ Identity is true for all permissable values of x

method 2 ... work on both sides until you reach equality on both sides at any point.

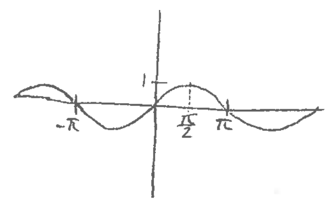
LS	RS
$= \frac{\cos x}{1-\sin x} \times \frac{1+\sin x}{1+\sin x}$	$= \frac{1+\sin x}{\cos x} \times \frac{\cos x}{\cos x}$
$= \frac{\cos x(1+\sin x)}{1-\sin^2 x}$	$= \frac{\cos x(1+\sin x)}{\cos^2 x}$
	$= \frac{\cos x(1+\sin x)}{1-\sin^2 x}$

$\because LS = RS$

$\therefore$ Identity is true for all permissable values of x

Restrictions: (in denominator)

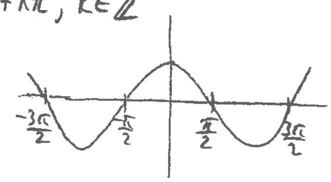
$1-\sin x \neq 0 \implies \sin x \neq 1 \implies x \neq \frac{\pi}{2}+2k\pi,\ k \in \mathbb{Z}$



$\cos x \neq 0 \implies x \neq \frac{\pi}{2}+k\pi,\ k\in\mathbb{Z}$

all together: $x \neq \frac{\pi}{2}+k\pi,\ k\in\mathbb{Z}$

(this takes care of both restrictions)

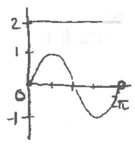


Ex 3 Solve algebraically over $[0, 2\pi)$: $1 - \cos^2 x = 3\sin x - 2$

Soln Ahhh! How do I factor with both $\cos x$ & $\sin x$ the way it's written here?!
I know — use a trig identity!!

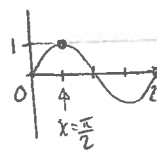
I know $1-\cos^2 x = \sin^2 x$ since $\sin^2 x + \cos^2 x = 1$ (take away $\sin^2 x$ from both sides),
So then...

$$1-\cos^2 x = 3\sin x - 2$$
$$\sin^2 x = 3\sin x - 2$$
$$\sin^2 x - 3\sin x + 2 = 0$$
$$\underbrace{\sin^2 x - 2\sin x}\ \underbrace{-\sin x + 2} = 0$$
$$\sin x(\sin x - 2) - (\sin x - 2) = 0$$
$$\underbrace{(\sin x - 2)}_{0}\ \text{or}\ \underbrace{(\sin x - 1)}_{0} = 0$$



$\therefore \sin x = 2$ impossible! $\sin x \neq 2$ $\therefore \sin x - 2 \neq 0$

or $\sin x = 1$ $\therefore x = \frac{\pi}{2}$ only solution on $[0, 2\pi)$



HWK: 6.1 p 296 # 4, 6
6.3 p 314 # (1-3)a, d ; 10b ; 11b, c
6.4 p 321 # 8, 10