Day 24 Review

① How long would it take for an investment to double at 8% per annum, compounded daily?

② Solve using exponent rules: a) $\dfrac{27^x}{9^{2x-1}} = 3^{x+4}$ b) $8(2x-1)^3 = 125$

Sol^n: ① Use formula $A(t) = P\left(1+\frac{i}{n}\right)^{nt}$ (see last day for detailed info).

Whatever P is, we need it to double, ie, $A(t) = 2P$

$i = 0.08$

$n = 365$

$\therefore \; 2\cancel{P} = \cancel{P}\left(1+\frac{0.08}{365}\right)^{365t}$

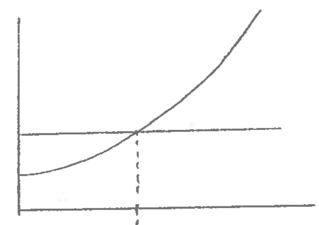
$2 \approx (1.000219178)^{365t}$

Once we learn about logarithms, we can make short work of this kind of question, but for now, let's use our graphing calculator...

$Y1 = 2$

$Y2 = 1.000219178 \wedge (365X)$

$\boxed{\text{INTERSECT}}$



$X = 8.6652892$

Therefore, it takes about 8.7 years.

Possible window:

	min	max
X	0	20
Y	0	5

-or-

take decimal part: $0.6652892 \text{ years} \times \frac{365 \text{ days}}{1 \text{ year}}$

≈ 242.830558 days

≈ 243 days

$\therefore$ It takes 8 years and 243 days to double ☺

② a) $\dfrac{27^x}{9^{2x-1}} = 3^{x+4}$

$\dfrac{3^{3x}}{3^{2(2x-1)}} = 3^{x+4}$

$3^{3x-2(2x-1)} = 3^{x+4}$

$3^{3x-4x+2} = 3^{x+4}$

$3^{-x+2} = 3^{x+4}$

like bases! equate exponents!

$\therefore -x+2 = x+4$

$-2 = 2x$

$\therefore x = -1$

b) $8(2x-1)^3 = 125$

method I:

$2^3(2x-1)^3 = 5^3$

take cubed root of both sides:

$\left[2^3(2x-1)^3\right]^{\frac{1}{3}} = \left(5^3\right)^{\frac{1}{3}}$

$2^{\frac{3}{3}}(2x-1)^{\frac{3}{3}} = 5^{\frac{3}{3}}$

$2(2x-1) = 5$

$4x-2 = 5$

$4x = 7$

$x = \frac{7}{4}$

method II:

$(2x-1)^3 = \frac{125}{8}$

$(2x-1)^3 = \left(\frac{5}{2}\right)^3$

like exponents means bases must be equal!

If exp is even, then base can be +/− ☺

$\therefore 2x-1 = \frac{5}{2}$

$4x-2 = 5$ (multiplied both sides by 2)

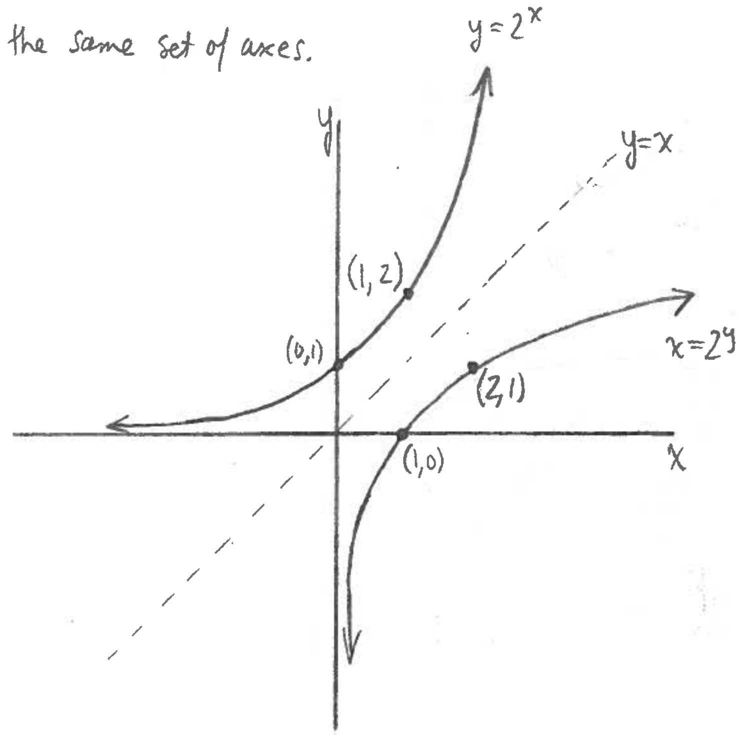
$4x = 7$

$x = \frac{7}{4}$

ch8 : Logarithmic Functions

8.1 – Understanding Logarithms

Sketch $y = 2^x$ and its inverse $x = 2^y$ on the same set of axes.



Until now, if we know the value of x, we could solve equations like $y = 2^x$ algebraically but not $x = 2^y$.

Introducing logarithms to save the day!

$$y = \log_2 x \iff x = 2^y$$

That is $y = \log_2 x$ and $x = 2^y$ are equivalent.
In fact, this is the definition of a logarithm:

In general... $\boxed{y = \log_b x \iff x = b^y}$ where b is the base

"logarithmic form" "exponential form"

Because our number system is base 10, $y = \log_{10} x$ is so common, we simply write $y = \log x$, and we call this the common logarithm, which is base 10 (i.e., $b = 10$)

☆ The most important concept to understand in this unit is being able to convert back & forth between logarithmic form & exponential form...

ex $y = \log_2 8$ means $8 = 2^y$ (same base) & $81 = 3^x$ means $x = \log_3 81$ (same base)

Since $y = \log_b x \iff x = b^y$, if we switch x & y for the exponential form, we get these inverses of one another: $\boxed{y = \log_b x \quad \& \quad y = b^x}$

inverses of each other

Dare we compare ☺ ...

log form	exp form	value of y
$y = \log_2 16$	$16 = 2^y$	4
$y = \log_2 \frac{1}{2}$	$\frac{1}{2} = 2^{-1}$	-1
$y = \log_{10} 1000$	$1000 = 10^y$	3
$y = \log_{10} 0.01$	$0.01 = 10^y$	-2
$y = \log_b b^n$	$b^n = b^y$	n

log identity: $\boxed{\log_b b^n = n}$

Ex1 Evaluate

a) $\log_2 32$

definition

let $x = \log_2 32$

Then $32 = 2^x$

$2^5 = 2^x$

$\therefore x = 5$

$\therefore \log_2 32 = 5$

useful when doing proofs...

above identity

$\log_2 2^5 = 5$

fast way!

b) $\log_9 \sqrt[5]{81}$

$= \log_9 81^{\frac{1}{5}}$

$= \log_9 (9^2)^{\frac{1}{3}}$

$= \log_9 9^{\frac{2}{3}}$

$= \frac{2}{3}$

c) $\log 10000$

$= \log_{10} 10^4$

$= 4$

d) $\log_3 9\sqrt{3}$

$= \log_3 3^2 \cdot 3^{\frac{1}{2}}$

$= \log_3 3^{2+\frac{1}{2}}$

$= \log_3 3^{2\frac{1}{2}}$

$= 2\frac{1}{2}$ or $\frac{5}{2}$

Other identities that will prove useful... (when $b>0$, $b \neq 1$) Why? Try to graph $y=(-2)^x$ and $y = 1^x$ and then inverses ☺

$\log_b 1 = 0$ Proof: let $\log_b 1 = x \xRightarrow[\text{form}]{\text{exp}} 1 = b^x \Rightarrow b^0 = b^x \Rightarrow x = 0 \quad \therefore \log_b 1 = 0$

$\log_b b = 1$ Proof: let $\log_b b = x \xRightarrow[\text{form}]{\text{exp}} b = b^x \Rightarrow b^1 = b^x \Rightarrow x = 1 \quad \therefore \log_b b = 1$

$b^{\log_b x} = x$ Proof: let $y = \log_b x \xRightarrow[\text{form}]{\text{exp}} x = b^y \Rightarrow x = b^{(\log_b x)}$

-OR- let $b^{\log_b x} = y \xRightarrow[\text{form}]{\text{log}} \log_b y = \log_b x \Rightarrow y = x \quad \therefore b^{\log_b x} = x$

Converting between log & exp forms are foundational in proofs!

Ex 3 Determine the value of x.

a) $\log_4 x = -2$

b) $\log_{16} x = -\frac{1}{4}$

c) $\log_x 9 = \frac{2}{3}$

Sol'n:

a) In exp form: $x = 4^{-2}$
$= \frac{1}{4^2}$
$= \frac{1}{16}$

b) In exp form: $x = 16^{-\frac{1}{4}}$
$= \frac{1}{16^{1/4}}$
$= \frac{1}{2^{4 \cdot \frac{1}{4}}}$
$= \frac{1}{2}$

c) In exp form: $9 = x^{\frac{2}{3}}$
$(9)^{\frac{3}{2}} = \left(x^{\frac{2}{3}}\right)^{\frac{3}{2}}$
$3^{2 \cdot \frac{3}{2}} = x^1$
$3^3 = x$
$\therefore x = 27$

Ex 4 State the inverse of $f(x) = \left(\frac{1}{2}\right)^x$. Then graph $y = f(x)$ & $x = f(y)$ on the same set of axes. Identify the domain & range, x + y intercepts (if they exist), and equations of any asymptotes of $x = f(y)$.

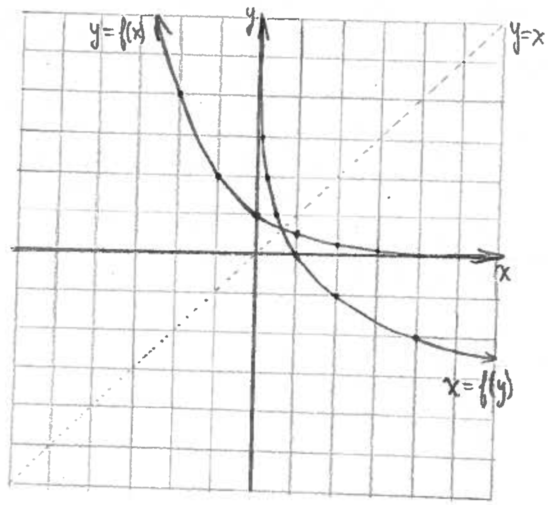
Sol'n: let $y = \left(\frac{1}{2}\right)^x$

In log form this is: $x = \log_{0.5} y$

So the inverse is: $y = \log_{0.5} x$
(switch $x \leftrightarrow y$)

$\therefore f(x) = \left(\frac{1}{2}\right)^x$ & $f^{-1}(x) = \log_{0.5} x$

Graph to see characteristics...



for $x = f(y)$,

D: $\{x \mid x > 0, x \in \mathbb{R}\}$

R: $\{y \mid y \in \mathbb{R}\}$

x-int: 1

y-int: none

vertical asymptote: $x = 0$

$y = f(x)$ has the opposite!!

Ex 5 The Richter Scale is based on a logarithmic scale. A magnitude of 6 on this scale is 10 times greater than a magnitude of 5. How many times more intense is an earthquake that is 9.5 compared to one that is 8.1 on the Richter Scale?

Sol'n: log scales are used when phenomena grow exponentially in order to scale down very large numbers. In the case of earthquakes, it is the amplitude of the tremour (ground wave) that is scaled down.

let A_1 be the amplitude of the earthquake of magnitude 9.5
and A_2 be the amplitude of the earthquake of magnitude 8.1

note: Richter scale is actually a ratio. This method here is a simplification & generalizes comparisons between logarithmic readings.

Then $9.5 = \log A_1 \Rightarrow A_1 = 10^{9.5}$
and $8.1 = \log A_2 \Rightarrow A_2 = 10^{8.1}$

Compare: $\frac{10^{9.5}}{10^{8.1}} = 10^{9.5-8.1} = 10^{1.4} \approx 25$

Thus, an earthquake of magnitude 9.5 is about 25 times more intense than one of 8.1.

HWK: p380-381 # 2-4, 7-9, 12, 16