Day 25 Review

① Solve for x: a) $x = \log_3 \sqrt{243}$ b) $\log_x 16 = -2$ c) $\log_{0.25} x = \frac{3}{2}$

② Write the inverse of $f(x) = 5^{x-1}$

③ The pH of a solution is given by $pH = -\log[H^+]$, where $[H^+]$ is the concentration of hydrogen ions in moles / litre.
The pH of normal skin is between 4.5 & 6.0 (varies with age).
How much more acidic is a pH of 4.5 than a pH of 6.0?

Sol^n: ① a)
$$x = \log_3\sqrt{243} = \log_3 (3^5)^{\frac{1}{2}} = \log_3 3^{5/2} = \frac{5}{2}$$

b) $\log_x 16 = -2$

exp form: $16 = x^{-2}$
$$4^2 = x^{-2}$$
$$\left(\frac{1}{4}\right)^{-2} = x^{-2}$$
$$\therefore x = \frac{1}{4}$$

c) $\log_{0.25} x = \frac{3}{2}$

exp form: $x = 0.25^{3/2}$
$$= \sqrt{\left(\frac{1}{4}\right)}^{3} = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

② let $y = 5^{x-1}$. This is equivalent to $\log_5 y = x - 1$

Switch $x \leftrightarrow y$ for inverse: $\log_5 x = y - 1$

$\therefore y = \log_5 x - 1$ or $f^{-1}(x) = \log_5(x) - 1$

③ pH of 4.5: $4.5 = -\log[H^+] \Rightarrow [H^+] = 10^{-4.5}$

pH of 6.0: $6.0 = -\log[H^+] \Rightarrow [H^+] = 10^{-6.0}$

Changed to exp form, as usual ☺

To see how many times more acidic pH of 4.5 is than 6.0, divide:

$$\frac{10^{-4.5}}{10^{-6.0}} = 10^{-4.5-(-6.0)} = 10^{1.5} \approx 31.6227\ldots$$

So pH 4.5 is about 32 times more acidic than pH of 6.0.

8.2 Transformations of Logarithmic Functions

By now you're an expert with transformations (or should be ☺), so let's jump right in...

W.r.t. $y = \log_b x$ ...

$$y = A \log_{b} \left(B(x+C)\right) + D$$

base

vertical translation by D units
- up if $D>0$
- down if $D<0$

vertical stretch by factor $|A|$ about x-axis. (Reflection in x-axis if $A<0$)

horizontal stretch by factor $\left|\frac{1}{B}\right|$ about y-axis. (reflection in y-axis if $B<0$)

horizontal translation by C units...
- left if $C>0$
- right if $C<0$

* Can you predict which parameters affect domain & range? (see example)

Ex1 a) Use transformations to sketch the graph of $y = -\log_2(2x+6) + 2$

b) Identify domain & range, equation of asymptote, x- & y-intercepts (if they exist)

Soln: Let's use mapping notation ☺

$y = \log_2 x \longrightarrow y = -\log_2(2(x+3)) + 2$

use exp form: $x = 2^y$

*easiest to start w/ y-values

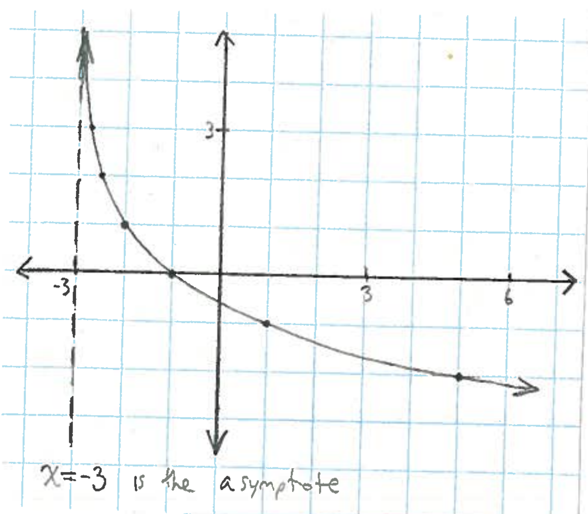
x	y	x	y
$2^{-1} = \frac{1}{2}$	-1	$\frac{1}{2}(\frac{1}{2}) - 3 = -2.75$	$-(-1)+2 = 3$
$2^0 = 1$	0	$\frac{1}{2}(1) - 3 = -2.5$	$-(0)+2 = 2$
$2^1 = 2$	1	$\frac{1}{2}(2) - 3 = -2$	$-(1)+2 = 1$
$2^2 = 4$	2	$\frac{1}{2}(4) - 3 = -1$	$-(2)+2 = 0$
$2^3 = 8$	3	$\frac{1}{2}(8) - 3 = 1$	$-(3)+2 = -1$
$2^4 = 16$	4	$\frac{1}{2}(16) - 3 = 5$	$-(4)+2 = -2$

$$(x, y) \longrightarrow \left(\tfrac{1}{2}x - 3,\ -y + 2\right)$$

* notice only parameter "C" affects domain & that range is always $\mathbb{R}$

Domain: $\{x \mid x > -3, x \in \mathbb{R}\}$

Range: $\{y \mid y \in \mathbb{R}\}$ ← always w/ logs ☺



x-int is -1 graphically or...

set $y=0 \implies 0 = -\log_2(2x+6)+2$

$\log_2(2x+6) = 2$

$2^2 = 2x+6$

$4 = 2x+6$

$-2 = 2x$

$x = -1$

y-int — set $x=0 \implies y = -\log_2(2(0)+6)+2$

$y = -\log_2 6 + 2$

$\log_2 6 = 2 - y$

exp form: $2^{2-y} = 6$

graphing calc: $\longrightarrow Y_1 = 2\wedge(2-X)$, $Y_2 = 6$

$\therefore y \approx -0.585$ intersect

HWK: p390 #4, 5, 6a, 9, 10; challenge: 15

8.3 Laws of Logarithms

Product Law: $\log_b MN = \log_b M + \log_b N$

Quotient Law: $\log_b \frac{M}{N} = \log_b M - \log_b N$

Power Law: $\log_b M^P = P \log_b M$

As most log proofs, convert to exponential form, apply exponent rules, then convert back to log form — TRY IT!
(or look in textbook p 394/5)

We will prove the product law — do the others for homework ☺

let $x = \log_b M$ & $y = \log_b N$

then $M = b^x$ & $N = b^y$ (change to exp form)

$\therefore MN = b^x \cdot b^y$

$MN = b^{x+y} \xrightarrow{\text{exp form}} \log_b MN = x + y = \log_b M + \log_b N$

Ex 1 Write each expression in terms of individual logarithms of x, y, z.

a) $\log_3 \frac{\sqrt{xy}}{z^2}$

$= \log_3 \frac{(xy)^{\frac{1}{2}}}{z^2}$

$= \log_3 \frac{x^{\frac{1}{2}} y^{\frac{1}{2}}}{z^2}$

$= \log_3 x^{\frac{1}{2}} + \log_3 y^{\frac{1}{2}} - \log_3 z^2$

$= \frac{1}{2} \log_3 + \frac{1}{2} \log_3 y - 2 \log_3 z$

b) $\log_2 \frac{8\sqrt{z}}{x^2 y}$

$= \log_2 8\sqrt{z} - \log_2 x^2 y$

$= \log_2 8 + \log_2 \sqrt{z} - (\log_2 x^2 + \log_2 y)$

$= \log_2 2^3 + \log_2 z^{\frac{1}{2}} - \log_2 x^2 - \log_2 y$

$= 3 + \frac{1}{2} \log_2 z - 2 \log_2 x - \log_2 y$

Ex 2 Evaluate: $2 \log_3 6 - \frac{1}{2} \log_3 64 + \log_3 2$

$= \log_3 6^2 - \log_3 64^{\frac{1}{2}} + \log_3 2$

$= (\log_3 36 + \log_3 2) - \log_3 8$

$= \log_3 \frac{36 \cdot \cancel{2}^1}{\cancel{8}_4}$

$= \log_3 9$

$= \log_3 3^2$

$= 2$

Ex 3 Write as a single logarithm in simplest form. State any restrictions.

a) $\log_5(2x-2) - \log_5(x^2+2x-3)$

b) $1 + \log_2(x^2-9) - \log_2(x^2-x-6)$

Sol'n: a) $\log_5(2x-2) - \log_5(x^2+2x-3)$

$= \log_5 \dfrac{2x-2}{x^2+2x-3}$

$= \log_5 \dfrac{2\cancel{(x-1)}}{\cancel{(x-1)}(x+3)}$

$= \log_5 \dfrac{2}{x+3} \longrightarrow x > -3$

but not all we consider...

Restrictions:

* must consider original restrictions:

$2x-2>0$ & $x^2+2x-3>0$

$x>1$ $\quad (x+3)(x-1)>0$

(see below)

3 intervals to consider: $x<-3$, $-3<x<1$, $x>1$

test a pt in each to determine restrictions on x:

in $(-\infty,-3)$: $(-)(-)>0$ ✓

in $(-3,1)$: $(+)(-)<0$ ✗

in $(1,\infty)$: $(+)(+)>0$ ✓

All together $x>1$ & $x<-3$ + $x>1$

$\Rightarrow x>1$ overall (limiting restriction)

* pleases all logs in question ☺

b) $1 + \log_2(x^2-9) - \log_2(x^2-x-6)$

$= \log_2 2^1 + \log_2 (x-3)(x+3) - \log_2 (x-3)(x+2)$

$= \log_2 \dfrac{2\cancel{(x-3)}(x+3)}{\cancel{(x-3)}(x+2)}$

$= \log_2 \dfrac{2x+6}{x+2} \longrightarrow$ like above, let's consider original log restrictions — they tell the full story ☺

restrictions...

$x^2-9>0$

$(x+3)(x-3)>0$

3 intervals to consider:

$x\in(-\infty,-3)$: $(-)(-)>0$ ✓

$x\in(-3,3)$: $(+)(-)<0$ ✗

$x\in(3,\infty)$: $(+)(+)>0$ ✓

$\therefore x\in(-\infty,-3)\cup(3,\infty)$

"Union" of sets (like adding for sets)

$x^2-x-6>0$

$(x+2)(x-3)>0$

3 intervals to consider:

$x\in(-\infty,-2)$: $(-)(-)>0$ ✓

$x\in(-2,3)$: $(+)(-)<0$ ✗

$x\in(3,\infty)$: $(+)(+)>0$ ✓

$\therefore x\in(-\infty,-2)\cup(3,\infty)$

All together, we consider the intersection of the allowable intervals; i.e, the intervals that all logs in question are defined for:

$[(-\infty,-3)\cup(3,\infty)] \cap [(-\infty,-2)\cup(3,\infty)]$

"intersection" of sets

$= (-\infty,-3)\cup(3,\infty)$

$\therefore$ overall, $x<-3$ or $x>3$

(can also write as $|x|>3$)

HWK: p 400-402 # (1-3)ac, 6, 9c, 11a, 13, 14, 16

Challenge: 19ab