Day 26

① The pt $(8,1)$ is on the graph of $y=\log_b x$. Determine the equation of a transformed function, with the same base, that passes through $(8,3)$.

② If $x=\log_2 3$ and $y=\log_2 5$, express $\log_2 45$ in terms of x and y.

③ Express $\log_4 \frac{2\sqrt{xy}}{x^2 z}$ as individual logarithms of x, y, z in simplest form.

④ State the domain & range of: a) $\log_3(2x-5)+1$ b) $3^{1-x}+2$

Sol'n: ① There are many possible answers here.

First, if $(8,1)$ is on $y=\log_b x$, then $1=\log_b 8$ means $b=8$

A number of transformations could take us from $(8,1)$ to $(8,3)$, including (but not limited to):

$y=3\log_8 x$ $y=\log_8 x+3$ $y=2\log_8 x+1$

②
$$\begin{aligned}\log_2 45 &= \log_2 9\times 5\\ &= \log_2 3^2\times 5\\ &= \log_2 3^2+\log_2 5\\ &= 2\log_2 3+\log_2 5\\ &= 2x+y \quad \text{(since } x=\log_2 3 \text{ + } y=\log_2 5\text{)}\end{aligned}$$

③
$$\begin{aligned}\log_4 \frac{2\sqrt{xy}}{x^2 z} &= \log_4 \frac{2x^{\frac{1}{2}}y^{\frac{1}{2}}}{x^2 z}\\ &= \log_4 \frac{2x^{-\frac{3}{2}}y^{\frac{1}{2}}}{z}\\ &= \log_4 2+\log_4 x^{-\frac{3}{2}}+\log_4 y^{\frac{1}{2}}+\log_4 z\\ &= \log_4 4^{\frac{1}{2}}-\frac{3}{2}\log_4 x+\frac{1}{2}\log_4 y+\log_4 z\\ &= \frac{1}{2}-\frac{3}{2}\log_4 x+\frac{1}{2}\log_4 y+\log_4 z\end{aligned}$$

④ a) $\log_3 \underbrace{(2x-5)}_{>0}+1$

$2x-5>0 \Rightarrow x>2.5$

D: $\{x \mid x>2.5,\ x\in\mathbb{R}\}$

R: $\{y \mid y\in\mathbb{R}\}$ (true for all logs)

b) $\underbrace{\underbrace{3^{1-x}}_{>0}+2}_{\therefore\ >2}$

D: $\{x \mid x\in\mathbb{R}\}$ (true for all exp eq'ns)

R: $\{y \mid y>2,\ y\in\mathbb{R}\}$

8.4 Logarithmic and Exponential Equations

This section puts it all together — using logs instead of trial & error or graphical methods ☺

Ex1 Solve. Round your answer to the nearest hundredth.

a) $2^x = 783$ b) $5(3^{4x+1}) = 810$ c) $6^{3x+1} = 8^{x+3}$

Sol'n: Take the log of both sides of the equation and apply log laws...

a)
$$\log(2^x) = \log(783)$$
$$\frac{x\log 2}{\log 2} = \frac{\log 783}{\log 2}$$
$$x = \frac{\log 783}{\log 2}$$
$$x \approx 9.61$$

b)
$$\frac{5(3^{4x+1})}{5} = \frac{810}{5}$$
$$3^{4x+1} = 162$$
$$\frac{3^{4x}\cdot 3}{3} = \frac{162}{3}$$
$$3^{4x} = 54$$
$$\log(3^{4x}) = \log(54)$$
$$\frac{4x\log 3}{4\log 3} = \frac{\log 54}{4\log 3}$$
$$x = \frac{\log 54}{4\log 3}$$
$$x \approx 0.91$$

c)
$$\log(6^{3x+1}) = \log(8^{x+3})$$
$$(3x+1)\log 6 = (x+3)\log 8$$
$$3x\log 6 + \log 6 = x\log 8 + 3\log 8$$
$$3x\log 6 - x\log 8 = 3\log 8 - \log 6$$
$$x(3\log 6 - \log 8) = 3\log 8 - \log 6$$
$$x = \frac{3\log 8 - \log 6}{3\log 6 - \log 8}$$
$$x \approx 1.35$$

Ex2 Determine the value of $\log_3 50$ to 2 decimal places.

let $x = \log_3 50$ and write in exponential form:

$$3^x = 50$$

Then $\log(3^x) = \log(50)$

$$x\log 3 = \log 50$$
$$x = \frac{\log 50}{\log 3} \approx 3.56 \qquad \therefore \log_3 50 = \frac{\log 50}{\log 3} \approx 3.56$$

In this same way, we can derive the change of base formula:

$$\boxed{\log_b x = \frac{\log_a x}{\log_a b}}$$

where a, b are positive real numbers other than 1

*TRY PROVING THIS IN THE SAME WAY WE DID EX 2

Just as $b^a = b^c$ implies $a = c$ (where $b \neq 0, 1$),

Similarly $\log_b a = \log_b c$ implies $a = c$ (where $b > 0$, $b \neq 1$; $a, c > 0$)

We will use the latter principal to solve logarithmic equations...

Ex 3 Solve: a) $\log_7 x + \log_7 4 = \log_7 12$ b) $\log_2(x-6) = 3 - \log_2(x-4)$

Sol^n: a) $\log_7 x + \log_7 4 = \log_7 12$

$$\log_7 4x = \log_7 12$$

same bases

$$\therefore 4x = 12$$

$$x = 3$$

b) $\log_2(x-6) = 3 - \log_2(x-4)$

$$\log_2(x-6) + \log_2(x-4) = 3$$

$$\log_2(x-6)(x-4) = 3$$

$$\log_2(x^2-10x+24) = 3$$

Three ways to think about this at this point...

i) Write in exponential form:

$$x^2 - 10x + 24 = 2^3$$

ii) logs of like bases:

$$\log_2(x^2-10x+24) = \log_2(2^3)$$

$$\therefore x^2 - 10x + 24 = 2^3$$

iii) exponents of base 2:

$$2^{\log_2(x^2-10x+24)} = 2^3$$

$$\therefore x^2 - 10x + 24 = 2^3$$

Using any method... $x^2 - 10x + 24 = 8$

$$x^2 - 10x + 16 = 0$$

$$(x-2)(x-8) = 0$$

$$\therefore x \neq 2 \text{ or } x = 8$$

extraneous root

$\therefore$ $x = 8$ is the only solution

Check restrictions on original log eq^n:

$$\log_2 \underbrace{(x-6)}_{x>6} = 3 - \log_2 \underbrace{(x-4)}_{x>4}$$

overall, $x > 6$

$\therefore x \neq 2$

Ex4 The half-life of C-14 is 5730 years. Because of its long half-life, C-14 is used by archeologists to determine the approximate age of fossil remains. Suppose a skeleton was found in an area where the amount of C-14 had diminished by 76.3%. How old is the skeleton?

Let $C(t)$ be the amount of C-14 after t years and C_0 be the initial amount of C-14.

You could have also let $C_0 = 1$ to indicate 100%

diminished by 76.3% means there's 23.7% remaining

$$\text{Then} \quad C(t) = C_0 \left(\tfrac{1}{2}\right)^{t/5730}$$

$$0.237 \cancel{C_0} = \cancel{C_0} \left(\tfrac{1}{2}\right)^{t/5730}$$

$$0.237 = \left(\tfrac{1}{2}\right)^{t/5730}$$

$$\log(0.237) = \log\left(\left(\tfrac{1}{2}\right)^{t/5730}\right)$$

$$\frac{\log 0.237}{\log 0.5} = \frac{\frac{t}{5730} \log 0.5}{\log 0.5}$$

$$5730 \times \frac{\log 0.237}{\log 0.5} = \frac{t}{5730} \times 5730$$

$$\frac{5730 \log 0.237}{\log 0.5} = t$$

$$\therefore t \approx 11901$$

∴ The skeleton is about 11 900 years old.

HWK: p 412-415 #6-8, 11, 13-17

challenge: #20, 21