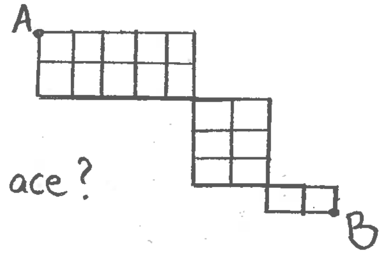
Day 30

Review

① Show three different ways to find the number of ways to go from pt A to B by only moving down and right.



② Suppose you are dealt 5 playing cards. How many different 5-card hands contain at least 1 ace?

p535 #16 ③ Verify that ${}_nC_r = {}_nC_{n-r}$.

Sol'n: ① method I: Permutations

DDRRRRR DDDRR DRR

$$\frac{7!}{2!\,5!} \times \frac{5!}{3!\,2!} \times \frac{3!}{1!\,2!} = 630$$

(21) (10) (3)

method II: Combinations

Notice in method I that the factors for each rectangular grid follow the combinations formula!

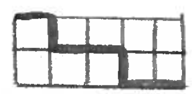
$${}_7C_5 \times {}_5C_2 \times {}_3C_2 = 630$$

(21) (10) (3)

or

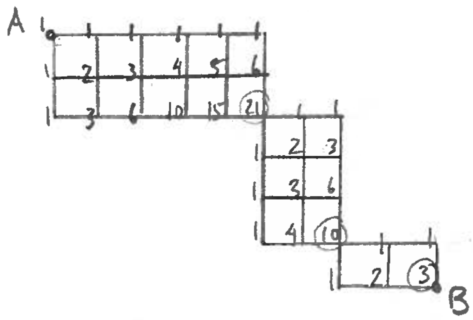
$${}_7C_2 \times {}_5C_3 \times {}_3C_1 = 630$$

Why? ex



any path you choose involves passing 7 blocks i.e. $n=7$ ∴ you either choose 5 right ($r=5$) or 2 down ($r=2$), giving ${}_7C_5$ or ${}_7C_2$ ☺

method III: Pascal's Triangle (preview!!)



$21 \times 10 \times 3 = 630$

same entries as in method I + II !! ☺

② method I: cases

"4 aces, choose 1; 48 other cards left, choose 4"

1 ace: ${}_4C_1 \cdot {}_{48}C_4 = 778\,320$

2 aces: ${}_4C_2 \cdot {}_{48}C_3 = 103\,776$

3 aces: ${}_4C_3 \cdot {}_{48}C_2 = 4\,512$

4 aces: ${}_4C_4 \cdot {}_{48}C_1 = 48$

886 656 hands

method II: subtract opposite from total

$${}_{52}C_5 - {}_4C_0 \cdot {}_{48}C_5 = 886\,656 \text{ hands}$$

(all possible hands) (no aces)

③ Show LS = RS (like trig identities ☺)

$LS = {}_nC_r = \dfrac{n!}{(n-r)!\,r!}$

$RS = {}_nC_{n-r} = \dfrac{n!}{(n-(n-r))!\,(n-r)!} = \dfrac{n!}{(n-n+r)!\,(n-r)!} = \dfrac{n!}{r!\,(n-r)!} = \dfrac{n!}{(n-r)!\,r!}$

∵ LS = RS ∴ statement true (for all $0 \le r \le n$; $n, r \in \mathbb{N}$)

11.3 The Binomial Theorem

Pascal's Triangle

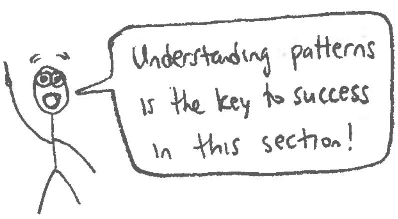
$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

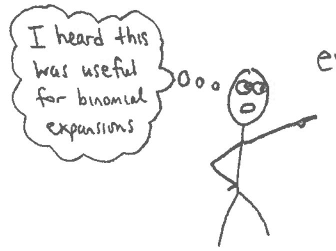
$3 + 1$
4

What patterns do you notice?
How are they related to combinations?

$n=0$	1	$=2^0$
$n=1$	$1+1$	$=2^1$
$n=2$	$1+2+1$	$=2^2$
$\vdots$	$\vdots$	$\vdots$

$$
\begin{array}{ccc}
 & {}_0C_0 & \\
{}_1C_0 & & {}_1C_1 \\
{}_2C_0 \quad & {}_2C_1 & \quad {}_2C_2 \\
 & \vdots &
\end{array}
$$



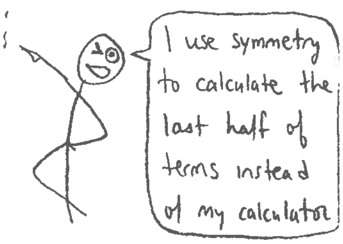


ex 6th row $(n=5)$:

$$
\begin{array}{cccccc}
1 & 5 & 10 & 10 & 5 & 1 \\
\shortparallel & \shortparallel & \shortparallel & \shortparallel & \shortparallel & \shortparallel \\
{}_5C_0 & {}_5C_1 & {}_5C_2 & {}_5C_3 & {}_5C_4 & {}_5C_5
\end{array}
$$

1st term always 1

2nd term always n



Binomial Theorem

Expand & Simplify:

$$(a+b)^0 = 1$$

$$(a+b)^1 = 1a + 1b$$

$$(a+b)^2 = 1a^2b^0 + 2a^1b^1 + 1a^0b^2$$

$$(a+b)^3 = 1a^3b^0 + 3a^2b^1 + 3a^1b^2 + 1a^0b^3$$

$$(a+b)^4 = \underset{{}_4C_0}{1}a^4b^0 + \underset{{}_4C_1}{4}a^3b^1 + \underset{{}_4C_2}{6}a^2b^2 + \underset{{}_4C_3}{4}a^1b^3 + \underset{{}_4C_4}{1}a^0b^4$$

$1 + 3 = 4 = n$

$n+1$ terms in expansion of $(a+b)^n$

Binomial Theorem:

$$(a+b)^n = {}_nC_0\, a^n b^0 + {}_nC_1\, a^{n-1}b^1 + {}_nC_2\, a^{n-2}b^2 + \cdots + {}_nC_{n-1}\, a^1 b^{n-1} + {}_nC_n\, a^0 b^n \;*$$

where $n \in \mathbb{N}$ and the general term $\boxed{t_{r+1} = {}_nC_r\, a^{n-r} b^r}$ with $r \leq n$, $r \in \mathbb{N}$

*don't memorize this formula — use patterns to understand it!

Ex1 Use the patterns in the binomial theorem and Pascal's Triangle to expand $(a+b)^7$

Use row $n=7$ (8th row) of Pascal's Triangle: 1 7 21 35 35 21 7 1

Then $(a+b)^7 = 1a^7b^0 + 7a^6b^1 + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7a^1b^6 + 1a^0b^7$

$= a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$

Ex2 Use the Binomial Theorem to expand $(2x-3)^4$ $\underbrace{\text{in descending powers of } x}$

descending powers of "a" value will be known as the "standard expansion of $(a+b)^n$"

for $(a+b)^n$

$a = 2x$

$b = -3$

$n = 4$

$$(2x-3)^4 = {}_4C_0(2x)^4(-3)^0 + {}_4C_1(2x)^3(-3)^1 + {}_4C_2(2x)^2(-3)^2 + {}_4C_3(2x)^1(-3)^3 + {}_4C_4(2x)^0(-3)^4$$

$$= 1\cdot 2^4x^4(1) + 4\cdot 2^3x^3(-3) + 6\cdot 2^2x^2(9) + 4\cdot 2x(-27) + 1\cdot 1\cdot 81$$

$$= 16x^4 - 96x^3 + 216x^2 - 216x + 81$$

Ex3 Determine the 4th term in the standard binomial expansion of $(3x-y^2)^5$

Use $t_{r+1} = {}_nC_r\, a^{n-r} b^r$

4th term $= t_4 \Rightarrow r+1=4$
$\Rightarrow r = 3$

and $n = 5$

$a = 3x$

$b = -y^2$

Then $t_4 = {}_5C_3\,(3x)^{5-3}(-y^2)^3$

$= 10\cdot 3^2x^2(-1)^3(y^2)^3$

$= \underbrace{10\cdot 3^2(-1)^3}_{\text{Coefficient}} \cdot x^2y^6$

Coefficient (some questions ask for only this)

$\therefore\ t_4 = -90x^2y^6$

Try: Find the numerical coefficient of x^7 in the expansion of $\left(3x^2 - \frac{1}{x}\right)^8$. (We'll discuss in next class' review)

HWK: pg 542-543 # $\underbrace{(4-7)b}_{\text{warm-up}}$, $\underbrace{9, 11, 13, 15}_{\text{Interesting patterns in Pascal's Triangle}}$, $\underbrace{17-20}_{\text{Great test questions!}}$