Day 8 Review

① a) Divide x^3+3x^2-5x+2 by $x-2$ using

i) long division,

ii) synthetic division,

and express your answer in the form $\frac{P(x)}{x-a} = Q(x) + \frac{R}{x-a}$

b) Identify any restrictions on the variable.

c) Verify your work.

② What is the remainder when $P(x) = 10x^4 - 11x^3 - 8x^2 + 7x + 9$ is divided by $x+1$?

Sol'n: ① i)

$$\begin{array}{r} x^2+5x+5 \\ x-2 \,\big)\, \overline{x^3+3x^2-5x+2} \\ \underline{x^3-2x^2} \\ 5x^2-5x+2 \\ \underline{5x^2-10x} \\ 5x+2 \\ \underline{5x-10} \\ 12 \end{array}$$

ii)

$$\begin{array}{c|cccc} -2 & 1 & 3 & -5 & 2 \\ - & & -2 & -10 & -10 \\ \hline \times & 1 & 5 & 5 & 12 \end{array}$$

Coefficients of $P(x)$ make up 1st row

remainder

$\therefore\ x^2+5x+5 \quad R12$

$\therefore \frac{x^3+3x^2-5x+2}{x-2} = (x^2+5x+5) + \left(\frac{12}{x-2}\right)$ for both i) + ii)

b) Since we're dividing by $x-2$, $x-2 \neq 0 \Rightarrow x \neq 2$

c) Check that $(x-a)Q(x) + R = P(x)$

$LS = (x-2)(x^2+5x+5) + 12$

$= x^3+5x^2+5x-2x^2-10x-10+12$

$= x^3+3x^2-5x+2$

$= RS$ ✓

② Use Remainder Theorem: If we divide $P(x)$ by $x-a$, then $P(a)$ is the remainder.

$x+1 = x-(-1)$

$\therefore a = -1$

so $P(-1) = 10(-1)^4 - 11(-1)^3 - 8(-1)^2 + 7(-1) + 9$

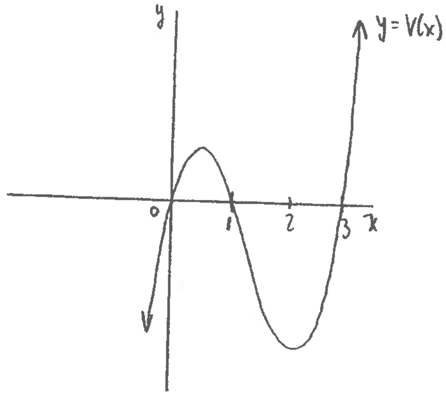
$= 10+11-8-7+9$

$= 15$

$\therefore \frac{P(x)}{x-a}$ gives a remainder of 15.

3.3 The Factor Theorem

Recall our example last class where we factored $V(x) = 2x^3 - 8x^2 + 6x$ to connect its factored form to its graph. We saw that the factored form showed us the roots of $V(x)$.



$$V(x) = 2x(x-1)(x-3)$$

roots: $x=0$ $x=1$ $x=3$

Now apply our Remainder Theorem for each factor.
That is, we use $a=0$, $a=1$, $a=3$.
Using the factored form it's easy to see $V(a)=0$.

This is true in general...

Factor Theorem: $x-a$ is a factor of polynomial in x, $P(x)$, if and only if $P(a)=0$

* If and only if means this is true both ways: $x-a$ is a factor of $P(x) \Longrightarrow P(a)=0$

-AND- $P(a)=0 \Longrightarrow x-a$ is a factor of $P(x)$

i.e. $x-a$ is a factor of $P(x) \Longleftrightarrow P(a)=0$

"if and only if" symbol
aka "iff"

Ex1 Which of the following binomials are factors of $P(x) = x^3 - 3x^2 - x + 3$?

a) $x-1$ b) $x+1$ c) $x-3$ d) $x+3$

Sol'n: Use Factor theorem: $x-a$ is factor of $P(x)$ iff $P(a)=0$

a) $x-1 \Rightarrow a=1$ $P(1) = (1)^3 - 3(1)^2 - (1) + 3 = 0 \Longrightarrow x-1$ is a factor of $P(x)$

b) $x+1 \Rightarrow a=-1$ $P(-1) = (-1)^3 - 3(-1)^2 - (-1) + 3 = 0 \Longrightarrow x+1$ is a factor of $P(x)$

c) $x-3 \Rightarrow a=3$ $P(3) = (3)^3 - 3(3)^2 - (3) + 3 = 0 \Longrightarrow x-3$ is a factor of $P(x)$

d) Since we found 3 factors of $P(x)$, we need not check $x+3$ (Indeed $P(-3) = -48 \neq 0$).
Why? Because $P(x)$ is of degree 3 and so has at most 3 roots!

In fact, we can see that $P(x)$ must factor to: $P(x) = (x-1)(x+1)(x-3)$

Notice in Ex1 that the values of a were factors of the constant term of $P(x)$.

In general ...

> **Integral Zero Theorem:** If $x = a$ is an integral zero of a polynomial, $P(x)$, with integral coefficients, then a is a factor of the constant term of $P(x)$.

integral (adj) means integer (n)

Ex 2 Fully factor $P(x) = 2x^3 - 5x^2 - 4x + 3$.

step 1: use Integral zero thm to find possible a values (i.e, integer values for zeros).

$P(x) = 2x^3 - 5x^2 - 4x + 3$ — constant term is 3, so possible factors are: $\pm 1, \pm 3$

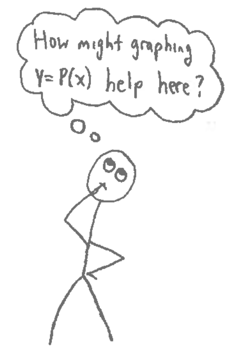
step 2: use factor thm to find at least one factor.

$P(1) = 2(1)^3 - 5(1)^2 - 4(1) + 3$
$= -4$

$P(1) \neq 0 \Rightarrow x - 1$ not a factor

$P(-1) = 2(-1)^3 - 5(-1)^2 - 4(-1) + 3$
$= 0$

$P(-1) = 0 \Rightarrow x + 1$ is a factor



note: we can continue with $P(3)$ + $P(-3)$ or move onto step 3.

Warning — we often can't rely on just Factor Thm to get all factors because some may repeat! (see next example)

step 3: use division and factoring.

$x+1$ factor ...

$$
\begin{array}{r|rrrr}
+1 & 2 & -5 & -4 & 3 \\
- & & 2 & -7 & 3 \\
\hline
 & 2 & -7 & 3 & 0
\end{array}
$$

— remainder better be zero ☺

$\therefore 2x^2 - 7x + 3$ remains

Factor: $P(x) = (x+1)(2x^2 - 7x + 3)$

$\therefore P(x) = (x+1)(2x-1)(x-3)$

aside: $2x^2 - 7x + 3$
$= 2x^2 - 1x - 6x + 3$
$= x(2x-1) - 3(2x-1)$
$= (2x-1)(x-3)$

$2 \times 3 = 6$
$\pm 1 \pm 2 \pm 3 \pm 6$
$(-1)(-6) = 6$ ✓
$-1 - 6 = -7$ ✓

(possible) step 4: repeat steps 2 + 3 as needed for higher degree polynomials.

Ex 3 Fully Factor $x^4-5x^3+2x^2+20x-24$.

Step 1: Use Integral Zero Theorem to find a values

let $P(x) = x^4-5x^3+2x^2+20x-24$

factors of -24: $\pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm10, \pm12, \pm24$

step 2: (factor Thm) $P(1) = (1)^4-5(1)^3+2(1)^2+20(1)-24$
$= -6$

$\because P(1) \neq 0 \Rightarrow x-1$ not a factor

easiest to start with low factors

$P(-1) = (-1)^4-5(-1)^3+2(-1)^2+20(-1)-24$
$= -36$

$\because P(-1) \neq 0 \Rightarrow x+1$ not a factor

$P(2) = (2)^4-5(2)^3+2(2)^2+20(2)-24$
$= 0$

$\because P(2) = 0 \Rightarrow x-2$ is a factor

step 3: divide & factor

$$\begin{array}{r|rrrrr} -2 & 1 & -5 & 2 & 20 & -24 \\ - & & -2 & 6 & 8 & -24 \\ \hline & 1 & -3 & -4 & 12 & 0 \end{array}$$

as you should get!

factor: $x^3-3x^2-4x+12$

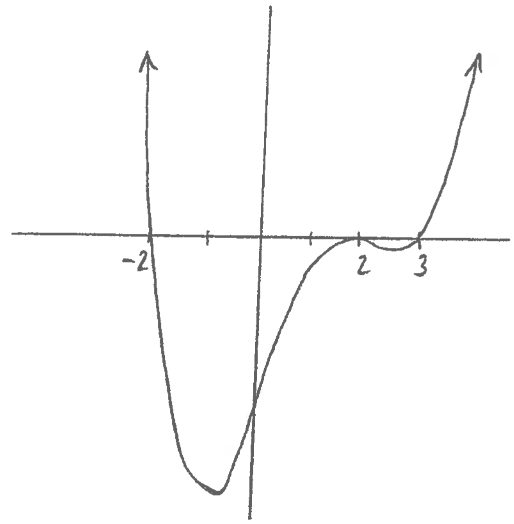
$$\begin{aligned} \therefore P(x) &= (x-2)(x^3-3x^2-4x+12) \\ &= (x-2)(x^2(x-3)-4(x-3)) \\ &= (x-2)(x-3)(x^2-4) \\ &= (x-2)(x-3)(x+2)(x-2) \end{aligned}$$

$\therefore P(x) = (x-2)^2(x-3)(x+2)$

difference of squares:
$x^2-a^2 = (x+a)(x-a)$

Alternative to steps 1 + 2:

Use graphing calculator
Y1 = X^4 - 5X^3 + 2X^2 + 20X - 24



We can see we have roots at $-2, 2, 3$
$\therefore$ $x+2$, $x-2$, $x-3$ are factors
So we can divide by 1 of these factors and then divide again by another to make the final factoring step easier.

Factoring difficult? Then do step 4... repeat steps 1 + 2 on $x^3-3x^2-4x+12$ instead of factoring.

$P(-2) = (-2)^3-3(-2)^2-4(-2)+12$
$= 0$

$\because P(-2) = 0 \Rightarrow x+2$ is a factor

$$\begin{array}{r|rrrr} +2 & 1 & -3 & -4 & 12 \\ - & & 2 & -10 & 12 \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

$\therefore x^2-5x+6$ is a factor

so $P(x) = (x-2)(x+2)(x^2-5x+6)$
$= (x-2)(x+2)(x-2)(x-3)$

$\therefore P(x) = (x-2)^2(x+2)(x-3)$

HWR: pg 133-135 # $\underbrace{1a, 2a, 3a, 4a}_{\text{warm-up}}$; 5-8, 13 challenge: 14, 15