Sequences & Series Hwk Booklet Solutions

① $8, 12, 18, 27, \ldots$

$r = \frac{t_{i+1}}{t_i}$ $(i \in \mathbb{N})$ ... in other words: $\boxed{\text{Common ratio} = \frac{\text{second term}}{\text{first term}}}$ (not on formula sheet ☹)

C

So $r = \frac{12}{8} = \frac{3}{2}$

② $a = t_1 = 30$ (1st term)

$r = 0.94$ b/c loses 6% (of original 100%), leaving 94%

A

So $30, 30(0.94), 30(0.94)^2, 30(0.94)^3, \ldots$

③ $5, 15, 45, \ldots$ ($\times 3$, $\times 3$, $\times 3$)

C

$r = 3$ & so

$a = 5$

$t_n = 885\,735$ ($n = ?$)

use $t_n = ar^{n-1}$

$885\,735 = 5(3)^{n-1}$

$177\,147 = 3^{n-1}$

$\log 177\,147 = \log(3)^{n-1}$

$\frac{\log 177147}{\log 3} = \frac{(n-1)\log(3)}{\log 3}$

$11 = n-1$

$\therefore n = 12$

OR $\frac{885\,735}{5} = \frac{5(3)^{n-1}}{5}$

$177\,147 = 3^{n-1}$

$3^{11} = 3^{n-1}$ guess & check exp

$\therefore 11 = n-1$

$\therefore n = 12$

④ $\sum_{k=1}^{n} 4(5)^{k-1}$

D

by definition, the sum of geometric series up to the nth term is given by:

$$S_n = \sum_{k=1}^{n} a(r)^{k-1}$$ (note $t_n = ar^{k-1}$ for geo seq)

So $S_n = \sum_{k=1}^{n} 4(5)^{k-1} \Longrightarrow a = 4, r = 5$

Using $S_n = \frac{a(1-r^n)}{1-r}$, we get

$$\sum_{k=1}^{n} 4(5)^{k-1} = \frac{a(1-r^n)}{1-r}$$
$$= \frac{(4)(1-5^n)}{1-5}$$
$$= \frac{-4(-1+5^n)}{-4}$$
$$= -1+5^n$$
$$= 5^n - 1$$

C ⑤

5m

up + down 2x
$0.60(5)$ m
$= 5(0.60) \times 2$
$= 10(0.60)$

up + down 2x
$0.60[0.60(5)]$ m
$= 5(0.60)^2 \times 2$
$= 10(0.60)^2$

$5 + \underbrace{10(0.60) + 10(0.60)^2 + 10(0.60)^3 + \dots}$

different

geo series with:
$a = 10(0.60)$
$r = 0.60$

total vertical distance $\Longrightarrow$ use $S_\infty = \dfrac{a}{1-r} = \dfrac{10(0.60)}{1-0.60} = 15$ m

don't forget the 5
$15\text{m} + 5\text{m}$
$= 20\text{m}$

C ⑥ common ratio of $3 - 1 + \frac{1}{3} - \frac{1}{9} + \dots + \frac{1}{243}$

$\times\left(-\frac{1}{3}\right) \quad \times\left(\frac{-1}{3}\right) \quad \times\left(\frac{-1}{3}\right) \Longrightarrow r = \frac{-1}{3}$

or use: $r = \dfrac{t_2}{t_1}$

$= \dfrac{-1}{3}$

can subtract any consecutive terms ☺

C ⑦ $8000, 4000, 2000, \dots$

$\times 0.5 \quad \times 0.5$

$t_9 = ?$
$a = 8000$
$r = \frac{1}{2}$

use $t_n = ar^{n-1}$
$t_9 = 8000\left(\frac{1}{2}\right)^8$
$= 8000\left(\frac{1}{256}\right)$
$= 31.25$

$(n-1 = 9-1 = 8)$

B ⑧ $S_5 = -328$
$r = -4$
$t_1 = a = ?$
$n = 5$

$S_n = \dfrac{a(1-r^n)}{1-r}$

$\therefore -328 = \dfrac{a(1-(-4)^5)}{1-(-4)}$

$\therefore -328 = \dfrac{a(1-(-1024))}{1+4}$

$-328 = \dfrac{a(1025)}{5}$

$\therefore a = -1.6$

C ⑨ $\displaystyle\sum_{k=1}^{\infty} 50\left(\frac{1}{4}\right)^k$

by definition, S_∞ or simply $S = \displaystyle\sum_{k=1}^{\infty} ar^{k-1}$ needs to be in this form

So $\displaystyle\sum_{k=1}^{\infty} 50\left(\frac{1}{4}\right)^k = 50\left(\frac{1}{4}\right)^1 + 50\left(\frac{1}{4}\right)^2 + 50\left(\frac{1}{4}\right)^3 + \dots$

$= 50\left(\frac{1}{4}\right)\left(1 + \left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^2 + \dots\right)$

$= 50\left(\frac{1}{4}\right)\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{4}\right)^{k-1}$ Yay!

$= 50\left(\frac{1}{4}\right)\left(\frac{4}{3}\right)$

$= \frac{50}{3}$

So use $S = \dfrac{a}{1-r}$

$a = 1$
$r = \frac{1}{4}$

$= \dfrac{1}{1-\frac{1}{4}}$

$= \dfrac{1}{\frac{3}{4}}$

$= \dfrac{4}{3}$

B (10) for a geo seq: $t_7 = 5x+2$ + $t_{10} = x-23$. $r=2$ $t_{10}=?$

use $t_n = ar^{n-1}$: $ar^6 = 5x+2$ ① $ar^9 = x-23$ ②

divide ② by ①: $\frac{\cancel{a}r^9}{\cancel{a}r^6} = \frac{x-23}{5x+2}$

$\therefore r^{9-6} = \frac{x-23}{5x+2}$

$\therefore r^3 = \frac{x-23}{5x+2}$

$(5x+2)r^3 = (x-23)$ (multiply both sides by $5x+2$)

use $r=3$, so $(2)^3 = 8 \Longrightarrow (5x+2)8 = (x-23)$

$40x + 16 = x - 23$

$\therefore 39x = -39$

$x = -1$ —— so $t_{10} = x-23 = -1-23 = -24$

B (11) $t_n = 5(-2)^{n-1} \Longrightarrow r = -2$ (since $t_n = ar^{n-1}$ for geo seq)

C (12) 1st term of expansion of $\sum_{k=2}^{8} 3(2^k) = 3(2^2) + 3(2^3) + 3(2^4) + \cdots + 3(2^8)$

$t_1 = 3(2^2) = 12$

C (13) $\frac{1}{128}, \frac{1}{32}, \frac{1}{8}, \ldots, 2048$ find n (# of terms)

$a = \frac{1}{128}$

$r = 4$ $\left(\frac{1}{8} \div \frac{1}{32} = \frac{1}{8} \times \frac{32}{1} = 4\right)$

$n = ?$

Can use $t_n = ar^{n-1}$

and $2048 = \frac{1}{128}(4)^{n-1}$

$262\,144 = 4^{n-1}$

$\log(262\,144) = \log(4)^{n-1}$

$\frac{\log(262144)}{\log 4} = \frac{(n-1)\cancel{\log(4)}}{\cancel{\log 4}}$

$\therefore n = 10$

B (14) $3 - 1 + \frac{1}{3} - \frac{1}{9} + \cdots$ $\times\left(-\frac{1}{3}\right)$

$a = 3$

$r = -\frac{1}{3}$

$\therefore S_\infty = \frac{a}{1-r} = \frac{3}{1-\left(-\frac{1}{3}\right)} = \frac{3}{\frac{4}{3}} \times \frac{\frac{3}{4}}{\frac{3}{4}} = \frac{9}{4}$

B (15) 9 days, so 9 terms; beginning with 270 mg; half the dose of previous day

$n=9$ $t_1=a=270$ $r=\frac{1}{2}$

total amount is the sum of the 9 terms: $S_n = \frac{a(1-r^n)}{1-r}$

$$S_9 = \frac{270\left(1-\left(\frac{1}{2}\right)^9\right)}{1-\frac{1}{2}}$$

$$\doteq \frac{270(0.998046875)}{0.5}$$

$$\doteq 538.95 \longrightarrow \text{so about } 539 \text{ mg.}$$

A (16) $-4, -1, -\frac{1}{4}, \ldots$

$\times\frac{1}{4}$ $\times\frac{1}{4}$

so $r = \frac{1}{4}$ (or use $\frac{t_2}{t_1} = \frac{-1}{-4} = \frac{1}{4} = r$)

C (17) $n=5$

$t_1 = a = \$38\,000$

$r = 1.02$ (2% extra means 102% of previous amount)

use $S_n = \frac{a(1-r^n)}{1-r}$ for $n=5$ $\longrightarrow$ $S_5 = \frac{38\,000(1-(1.02)^5)}{1-1.02}$

$\doteq \$197\,753.53$

C (18) $\sum_{k=3}^{7} 5(2)^k = 5(2)^3 + 5(2)^4 + 5(2)^5 + 5(2)^6 + 5(2)^7$

$= 5(8+16+32+64+128)$

$= 5(248)$

$= 1240$

C (19) $t_3 = 48$ & $t_6 = \frac{81}{4}$ ... so $t_1 = a = ?$

method 1: $ar^2 = 48$ ① $ar^5 = \frac{81}{4}$ ②

② ÷ ①: $\frac{\cancel{a}r^5}{\cancel{a}r^2} = \frac{\frac{81}{4}}{48}$

$\therefore r^{5-2} = \frac{81}{4} \times \frac{1}{48}$

$\sqrt[3]{r^3} = \sqrt[3]{\frac{27}{64}}$

$\therefore r = \frac{3}{4}$

use ①: $a\left(\frac{3}{4}\right)^2 = 48$

$a\left(\frac{9}{16}\right) = 48$

$a = 48 \times \frac{16}{9}$

$= \frac{16 \times 16}{3}$

$= \frac{256}{3}$

✱ use $n-2$ for each:

method 2: Change $t_3 = 48$ & $t_6 = \frac{81}{4}$ so find t_{-1}!

to $t_1 = 48$ to $t_4 = \frac{81}{4}$

$\therefore a = 48$

so $t_4 = ar^3 \Rightarrow 48r^3 = \frac{81}{4}$

$\Rightarrow r = \frac{3}{4}$

$\therefore t_{-1} = ar^{(-1)-1}$

$= 48\left(\frac{3}{4}\right)^{-2}$

$= \frac{48 \cdot 4^2}{3^2} = \frac{256}{3}$

D (20) $(x+1) + (x+1)^2 + (x+1)^3 + \ldots$

$r = (x+1)$ & we know S_∞ is finite if $|r|<1$ or $-1<r<1 \Rightarrow r>-1$ or $r<1$, $r\neq$

$\therefore |x+1|<1 \Longrightarrow x+1>-1$ or $x+1<1$, $x\neq-1$

$\therefore x>-2$ or $x<0$

$\therefore -2<x<0$, $x=-1$

nice question!! :)

B (21) $-64, 48, -36, \ldots$

$r = \frac{48}{-64} = \frac{-3}{4}$

C (22) $4, 8, 16, \ldots$ & $S_n = \frac{4(1-2^5)}{1-2} \Longrightarrow$ $a=4$, $r=2$, $n=5$

B (23) $\frac{1}{8}, \frac{1}{2}, 2, \ldots, 524\,288$ ($\times 4$, $\times 4$)

$a = \frac{1}{8}$

$r = \frac{1}{2} \div \frac{1}{8} = 4$

$t_n = 524\,288$

$n = ?$

use $t_n = ar^{n-1}$

$524\,288 = \frac{1}{8}(4)^{n-1}$

$\therefore 4\,194\,304 = 4^{n-1}$ ⟶ or use logs! $\log(4\,194\,304) = \log(4)^{n-1}$

$4^{11} = 4^{n-1}$

$\therefore n-1 = 11$

$n = 12$

$\frac{\log(4\,194\,304)}{\log 4} = \frac{(n-1)\log(4)}{\log 4}$

$n-1 = 11$

$n = 12$

C (24) $\sum_{k=1}^{\infty} 200(0.6)^{k-1}$ is equal to S_∞ where $a=200$ & $r=0.6$

So $S_\infty = \frac{a}{1-r} = \frac{200}{1-0.6} = 500$

B (25) $x, 4, 8x, \ldots$ $x=?$

$x\left(\frac{4}{x}\right)$ $x\left(\frac{8x}{4}\right)$

$r = \frac{t_2}{t_1}$ $r = \frac{t_3}{t_2}$

* $r = \frac{\text{any term}}{\text{term before}}$

So $\frac{4}{x} = \frac{8x}{4}$

both are r, so they must be equal!

another nice question :)

$\Rightarrow 8x^2 = 16$

$x^2 = 2$

$x = \pm\sqrt{2}$

C (26) $t_{10} = ?$ $\quad 2, 6, 18, 54, \ldots$ $\quad$ (×3, ×3, ×3)

$a = 2$

$r = 3$ $\qquad t_n = ar^{n-1} \Longrightarrow t_{10} = (2)(3)^{10-1} = 39\,366$

$n = 10$

C (27) $\displaystyle\sum_{k=4}^{9} 5(2)^k = 5(2)^4 + 5(2)^5 + 5(2)^6 + 5(2)^7 + 5(2)^8 + 5(2)^9$

$= 5(2^4 + 2^5 + 2^6 + 2^7 + 2^8 + 2^9)$

$= 5(16 + 32 + 64 + 128 + 256 + 572)$

$= 5(1008)$

$= 5040$

Another way... $\displaystyle\sum_{k=4}^{9} 5(2)^k = \sum_{k=5}^{10} 5(2)^{k-1}$

$\displaystyle = \sum_{k=1}^{10} 5(2)^{k-1} - \sum_{k=1}^{4} 5(2)^{k-1}$ $\qquad$ (why?)

$= S_{10} - S_4$

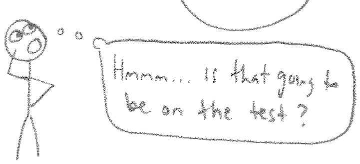
$= \dfrac{a(1-r^{10})}{1-r} - \dfrac{a(1-r^4)}{1-r}$, where $a = 5$ & $r = 2$

$= \dfrac{5(1-2^{10}) - 5(1-2^4)}{1-2}$

$= 5040$

useful with long series

ex $\displaystyle\sum_{k=10}^{100} 5(2)^{k-1}$



C (28) $r = 1.025$ $\left(\begin{array}{l}\text{since 2.5\% more of amount}\\ \text{gives 102.5\% of it}\end{array}\right)$

$a = t_1 = ?$ (need to find)

$n = 11$ (we begin at year zero)

$t_{11} = \$1267.28$

$t_n = ar^{n-1}$, t_n gives total after $n-1$ years (b/c $n=1$ is 1st invested at yr 0)

$t_{11} = a(1.025)^{10}$

$\therefore 1267.28 = a(1.025)^{10}$

$\therefore a \doteq \$990.00$ ← Initial investment.

OR use $A = P(1+i)^n$, $A = 1267.28$, $i = 0.025$, $n = 10$; find P

* not on formula sheet!

:) (29) $t_2 = -16$ + $t_7 = 512$. Find t_1

$t_2 = \boxed{ar = -16}$ ① $t_7 = \boxed{ar^6 = 512}$ ②

divide ② by ①: $\frac{ar^6}{ar} = \frac{512}{-16}$

$\therefore r^5 = -32$

$\therefore r = \sqrt[5]{-32}$

$r = -2$

use ①: $a(-2) = -16$

$\Rightarrow a = 8$

$\therefore t_1 = 8$

A (30) $a - 1 + \frac{1}{a} - \frac{1}{a^2} + \ldots$ $a > 1$

$\times\left(-\frac{1}{a}\right)$

$r = -\frac{1}{a}$

If not obvious, do $r = \frac{t_2}{t_1} = \frac{-1}{a}$

$$S_\infty = \frac{a}{1-r} = \frac{a}{1-\left(-\frac{1}{a}\right)} = \frac{a}{\frac{a}{a}+\frac{1}{a}} = \frac{a}{\left(\frac{a+1}{a}\right)} \times \frac{\left(\frac{a}{a+1}\right)}{\left(\frac{a}{a+1}\right)} = \frac{a^2}{a+1}$$

C (31) $$\sum_{k=2}^{6} 3(3)^{k-2} = \sum_{k=1}^{5} 3(3)^{k-1} = \frac{3(1-3^5)}{1-3} = 363$$

$\because S_6 = \sum_{k=1}^{5} 3(3)^{k-1} = \frac{a(1-r^5)}{1-r}$, $a = 3$ + $r = 3$

C (32) $r = 2$

$t_{12} = 16\,384$

$t_1 = a = ?$

$t_{12} = ar^{11} \Rightarrow 16\,384 = a(2)^{11}$

$\Rightarrow a = 8$

~~A~~ (33) $3, \frac{2}{5}, \frac{4}{75}, \ldots$

$r = \frac{2}{5} \div 3$

$= \frac{2}{5} \times \frac{1}{3}$

$= \frac{2}{15}$

D (34) $t_3 = 45$ + $t_6 = 1215$ $\quad t_1 = a = ?$

$$t_3 = \boxed{ar^2 = 45} \; ① \qquad t_6 = \boxed{ar^5 = 1215} \; ②$$

use ①: $a(\pm 3)^2 = 45$

$$\Rightarrow a = \frac{45}{9} = 5$$

divide ② by ①: $\dfrac{\cancel{a}r^5}{\cancel{a}r^2} = \dfrac{1215}{45}$

$$r^3 = 27$$

$$\therefore r = \sqrt[3]{27}$$

$$r = \pm 3$$

D (35) $\displaystyle\sum_{k=1}^{n} 4(5)^{k-1} \Longrightarrow S_n = \sum_{k=1}^{n} ar^{k-1}$ where $a = 4$, $r = 5$

Using $S_n = \dfrac{a(1-r^n)}{1-r}$

$$= \frac{4(1-5^n)}{1-5}$$

$$= \frac{4(1-5^n)}{-4}$$

$$= -(1-5^n)$$

$$= 5^n - 1$$

D (36) $y = 2^x$, y is # ancestors x generations ago

$2^1, 2^2, 2^3, \ldots$

2^1: 1 generation ago (2 parents) $n=1$

2^2: 2 generations ago (4 grand parents) $n=2$

So your total ancestors in the last n generations is given by:

$$2 + 2^2 + 2^3 + \cdots + 2^n$$

$$= 2(1 + 2^1 + 2^2 + \cdots + 2^{n-1})$$

$$= 2\left(\frac{a(1-r^n)}{1-r}\right) \qquad \because S_n = \sum_{k=1}^{n} ar^{n-1} = \frac{a(1-r^n)}{1-r}$$

where $a = 1$, $r = 2$

$$= 2\left(\frac{1(1-2^n)}{1-2}\right)$$

$$= 2\left(\frac{-(2^n-1)}{-1}\right)$$

$$= 2(2^n - 1)$$